Problem Set 8 Solutions

2:35PM in lecture, Tuesday, November 7, 2000 **Be sure to indicate your TA's name clearly on the solutions your turn in**.

Self-study

Read: Rosen Section 4.1 Self-study problems (not to be handed in): Exercises 1, 3, 5, 7, 9, 11, 15, 17

Reading: Lecture notes; Rosen 4.1, 5,3, 5.5, 5.6

Policy on collaboration If you collaborated with someone on the problem sets, you should mention their names on top of the problem set. If you did not collaborate with anyone, you should explicitly say so. As a reminder, the write-up of your solutions should be entirely your own.

Problem 1 In a survey of 270 college students, it is found that:

- 64 like brussels sprouts
- 94 like broccoli
- 58 like cauliflower
- 26 like both brussels sprouts and broccoli
- 28 like both brussels sprouts and cauliflower
- 22 like both broccoli and cauliflower
- 14 like all three vegetables

How many students do not like ANY of the aforementioned vegetables?

Solution: By the direct application of the Principle of Inclusion-Exclusion, the union of sprouts-lovers, cauliflower-lovers, and broccoli-lovers will be of size: $64 + 94 + 58 - 26 - 28 - 22 + 14 = 154$ Therefore, the remaining $270 - 154 = 116$ students don't like any of the three vegetables.

Problem 2

(a) In how many ways can the natural numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ be arranged in a sequence so that no number is in its original position?

Solution:

We need to determine the number of **derangements** of a set of 12 elements.

$$D_{12} = 12!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!} - \frac{1}{11!} + \frac{1}{12!}\right] = 176,214,841$$

(b) In how many ways can the natural numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ be arranged in a sequence so that no *even* number is in its original position?

Solution:

We can derive this answer by mimicking the derivation of the formula for the number of derangements, but worrying only about the even numbers. What we want is the size of the union of all those undesirable permutations that keep one or more of the even numbers in their place, so that we can subtract it from the sum of all permutations.

There are 12! permutations altogether. Let e be one of the 6 even numbers. The number of permutations in which e is in its original position is 11! (since e remains in its place while the other 11 numbers are free to move around however they like.)

So, there are $11!\binom{6}{1}$ permutations which force 1 of the even numbers stays in its place. Similarly, there are $10!\binom{6}{2}$ permutations which keep 2 of the even numbers in their original positions, $9!\binom{6}{3}$ permutations that force 3 of them to remain where they were, $8!\binom{6}{4}$ permutations that keep 4 even numbers fixed, $7!\binom{6}{5}$ permutations that guarantee that 5 even numbers will stay put, and finally $6!\binom{6}{6}$, that is 6! permutations in which all 6 even numbers will remain where they were.

We want to know how many permutations there are where **at least** one number remains in its place. The inclusion-exclusion principle comes to our rescue, and says that the size of the union of these undesirable permutations is:

$$11!\binom{6}{1} - 10!\binom{6}{2} + 9!\binom{6}{3} - 8!\binom{6}{4} + 7!\binom{6}{5} - 6!\binom{6}{6} = 191,751,120$$

There are 12! permutations in total, $191,751,120$ of which are undesirable, which leaves us with $12! - 191,751,120 = 287,250,480$ permutations where no number appears in its original position.

Problem 3 How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution:

There are 2^{10} bit strings of length 10. First, let's see how many of those contain five consecutive 0's. The sequence of the 5 0's can start at position $1, 2, 3, 4, 5$ or 6. In each case, the remaining 5 bits are allowed to be anything, so, as a "first approximation", we can say that there are $6 * 2^5$ different strings of consecutive 0's.

However, if we do this, we are *overcounting* since, for example, the string that has its first 8 bits equal to 0 and the last 2 equal to 1 (that is, 0000000011) will be counted as a valid string *four times* by our method: $\underline{00000}00011$, $0\underline{00000}0011$, $00\underline{00000}011$ and $000\underline{00000}11$. We need to eliminate these duplicates, which occur whenever we are dealing with strings that have more than 5 consecutive 0's.

We can do this by calculating the number of strings with 6 or more consecutive 0's using the same method as above. This will give us $5 * 2^4$ such strings, and for each sequence of 0's, we will be overcounting exactly one time less than we overcounted above: for example, going back to the string of 8 consecutive 0's that was counted 4 times above, we see that this method will count it exactly 3 times $\underline{000000}0011$, $0\underline{000000}011$, $00\underline{000000}11$. The same will hold for all strings of length ≥ 6 because the number of allowed starting positions for a sequence of 6 0's is exactly one less than the number of allowed starting positions for a sequence of 5 consecutive 0's. Thus, it suffices to subtract this number from the number we got before to get rid of duplicates. Therefore, the number of strings containing 5 or more consecutive 0's is

$$6 * 2^5 - 5 * 2^4 = 112$$

By symmetry, there are another 112 strings which contain five consecutive 1's. The question was to determine the number of strings that have either 5 0's or 5 1's. So, we are looking for the size of the union of those two sets (the bit strings with 5 0's and the bit strings with 5 1's). The intersection of the 2 sets (i.e, sequences that contain *both* 5 0's and 5 1's) contains 2 elements: 0000011111 and 1111100000. Therefore, by the inclusion-exclusion principle, the size of the union of the two sets is $112 + 112 - 2 = 222$

Problem 4

(a) Recall the double induction example from Lecture 3, which involved a function $f : \mathbb{N} \times \mathbb{N} \Rightarrow \mathbb{N}$ defined by: $f(n, 0) = 1$ for all n, $f(0, k) = 1$ for all k, and $f(n+1, k+1) = f(n+1, k) + f(n, k+1)$ for all n and k. We proved by double induction that $f(n, k) = \frac{(n+k)!}{n!k!}$ for all n and k.
Recall the definition of binomial coefficients: $\binom{a}{b} = \frac{a!}{(a-b)!b!}$. Express $f(n, k)$ in terms of binomial coefficients.

Solution:

Since

$$\frac{(n+k)!}{n!k!} = \frac{(n+k)!}{(n+k-k)!k!} = \binom{n+k}{k}$$

we see that $f(n, k) = \binom{n+k}{k}$

(b) What identity about binomial coefficients arises from what we proved in Lecture 3? Hint: how can you express $\binom{n+1}{k+1}$ in terms of smaller binomial coefficients?

Solution:

By direct substitution from part (a) we get:

$$\binom{n+k+2}{k+1} = \binom{n+k+1}{k+1} + \binom{n+k+1}{k}$$

This is usually written as:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

(c) Recall the definition of Fibonacci numbers:

$$F_0 = 0; F_1 = 1; \text{for every } n \geq 2, F_n = F_{n-2} + F_{n-1}$$

Prove that for all $n \geq 0$

$$F_{2n} = \sum_{i=0}^{n-1} \binom{2n-i-1}{i} \qquad \text{AND} \qquad F_{2n+1} = \sum_{i=0}^{n} \binom{2n-i}{i}$$

Hint: use induction, and the formula you derived above

Solution: We can prove this by induction on n. Let $P(n)$ be the statement that

$$F_{2n} = \sum_{i=0}^{n-1} \binom{2n-i-1}{i} \qquad \text{AND} \qquad F_{2n+1} = \sum_{i=0}^{n} \binom{2n-i}{i}$$

Base Case: For $n = 0$, $\sum_{i=0}^{-1} \binom{2n-i-1}{i} = 0$ since the summation basically has no terms. Indeed $F_0 = 0$ so the first half of what we need to show holds. $\sum_{i=0}^{0} \binom{2n-i}{i} = 1$ and $F_1 = 1$ so the second half also holds.

Induction Step: Assume $P(n)$. We want to prove $P(n+1)$. We need to show 2 things:

$$F_{2n+2} = \sum_{i=0}^{n} \binom{2n-i+1}{i} \qquad \text{and} \qquad F_{2n+3} = \sum_{i=0}^{n+1} \binom{2n-i+2}{i}$$

First, let's see what $\sum_{i=0}^{n} \binom{2n-i+1}{i}$ is equal to. The first term of the summation, i.e, when $i = 0$ is 1, therefore, we can write this as

$$1 + \sum_{i=1}^{n} \binom{2n-i+1}{i}$$

Now, using the identity from part (b), we can write this as

$$1 + \sum_{i=1}^{n} \left[\binom{2n-i}{i} + \binom{2n-i}{i-1} \right]$$

which is, of course, equal to

$$1 + \sum_{i=1}^{n} \binom{2n-i}{i} + \sum_{i=1}^{n} \binom{2n-i}{i-1}$$

We notice that if we let $i = 0$, $\binom{2n-i}{i} = 1$, so we can "absorb" that 1 into the first summation and write

$$\sum_{i=0}^{n} \binom{2n-i}{i} + \sum_{i=1}^{n} \binom{2n-i}{i-1}$$

Things are starting to look good since we recognize that first sum to be F_{2n+1}. In order to attack the second sum, we make a substitution: let $k = i - 1$ (which means, wherever we see i, we put $k + 1$. So the above becomes

$$\sum_{i=0}^{n} \binom{2n-i}{i} + \sum_{k=0}^{n-1} \binom{2n-k-1}{k}$$

From our induction hypothesis, this equals $F_{2n+1} + F_{2n}$ so this does indeed equal F_{2n+2}. (For those who are confused about what I proved: if you want to prove that $a = b$, you can just prove that $b = a$).

Now, let's see what $\sum_{i=0}^{n+1} \binom{2n-i+2}{i}$ is equal to. For $i = n+1$, $\binom{2n-i+2}{i} = \binom{n+1}{n+1} = 1$. For $i = 0$, $\binom{2n-i+2}{i} = 1$. Therefore, we can write this thing as

$$1 + 1 + \sum_{i=1}^{n} \binom{2n+-i+2}{i}$$

which, using the same identity again, can be written as

$$1 + 1 + \sum_{i=1}^{n} \left[\binom{2n-i+1}{i} + \binom{2n-i+1}{i-1} \right]$$

which is equal to

$$1 + 1 + \sum_{i=1}^{n} \binom{2n-i+1}{i} + \sum_{i=1}^{n} \binom{2n-i+1}{i-1}$$

Since for $i = 0$, $\binom{2n-i+1}{i} = 1$ and for $i = n+1$, $\binom{2n-i+1}{i-1} = \binom{n}{n} = 1$, we can "absorb" one of the 1's into the first sum and the other into the second sum and write

$$\sum_{i=0}^{n} \binom{2n-i+1}{i} + \sum_{i=1}^{n+1} \binom{2n-i+1}{i-1}$$

and making the substitution $i - 1 = k$ into the second sum, we get

$$\sum_{i=0}^{n} \binom{2n-i+1}{i} + \sum_{k=0}^{n} \binom{2n-k}{k}$$

By what we proved before, the first sum equals F_{2n+2} and by our induction hypothesis, the second sum equals F_{2n+1}. Therefore, $\sum_{i=0}^{n+1} \binom{2n-i+2}{i} = F_{2n+2} + F_{2n+1} = F_{2n+3}$ as desired.

Problem 5 On a set of n elements how many of the following are there?

(a) binary relations

Solution:

2^{n^2} since there are this many different $0-1$ matrices of order n.

(b) symmetric binary relations

Solution: The matrix associated with a symmetric relation is symmetric. Since there are $2^{n+\binom{n}{2}}$ different symmetric matrices there are an equal number of symmetric relations.

(c) reflexive binary relations

Solution: The matrix associated with a reflexive relation has 1's along the main diagonal . Hence the number of such matrices, and therefore reflexive relations, is 2^{n^2-n}.

(d) symmetric and reflexive binary relations

Solution: The matrix associated with a symmetric and reflexive relation is symmetric and has 1's along the main diagonal. Hence the number of such relations is $2^{\binom{n}{2}}$.

(e) symmetric or reflexive binary relations

Solution: The number of symmetric or reflexive relations = the number of reflexive relations + the number of symmetric relations $-$ the number of reflexive and symmetric relations $= 2^{n^2-n} + 2^{n+\binom{n}{2}} - 2^{\binom{n}{2}}$.