Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 38
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Tutorial 9 Problems

Problem 1 In lecture we demonstrated how to "guess" the fifth card in a poker hand when a collaborator reveals the other four cards. Describe a method for guessing two hidden cards in a hand of nine cards, when your collaborator reveals the other seven cards.

Solution:
There must be $\lceil 9/4 \rceil = 3$ cards with the same suit, so our collaborator chooses to hide two of them and then use the third one as the first card to be revealed. So this first revealed card fixes the suit of the two hidden cards; it will also be used as the origin for the offset position of the first hidden card. This first hidden card will in turn be used as the origin for the offset of the other hidden card. There are six cards to code the two offset positions. These suffice to code two offsets of size from one to six. That is, our collaborator can choose one of the $3! = 6$ orders in which to reveal the first three cards and thereby tell us the offset position of the first hidden card. Our collaborator can then choose the order of the final three cards to describe the offset position of the second hidden card from the first. Note that the first revealed card must be chosen so that both offsets are less ≤ 6; since the sum of the offsets between successive cars ordered in a cycle from Ace to King is 13, it is not possible for more than one offset between successive cards to exceed seven, so this is always possible.

Problem 2 In how many ways can $2n$ students be divided into n pairs?

Solution:

$$\frac{(2n)!}{n!2^n}$$

Consider all the ways of ordering the $2n$ students in a line. There are $(2n)!$ ways of doing this. Now group them into pairs based on the linear order, i.e. group the first and second students, group the third and fourth students,..., group the $(2n-1)$th and $2n$th students. Thus each linear order gives us a unique grouping into pairs. But observe that two different linear orders could give rise to the same grouping into pairs. So let us now count the number of linear orders that give rise to the same grouping into pairs. Given a grouping into pairs, to generate a linear order we need to order each pair (there are 2^n ways to do this) and then order the n pairs (there are $n!$ ways to do this). Hence $n!2^n$ linear orders give the same grouping into pairs. Thus the total number of ways $2n$ students can be divided into n pairs is $\frac{(2n)!}{n!2^n}$.

Problem 3 A positive integer is called *square-free* if it is not divisible by the square of any positive integer greater than 1. For example $35 = 5 \cdot 7$ is square-free but $18 = 2 \cdot 3^2$ is not.

1 is square-free. Use inclusion-exclusion to find the number of square-free positive integers strictly less than 151.

Solution:

We use inclusion-exclusion as done in lecture for the case of prime numbers. We first compute the number of positive integers less than 150 that are not square free.

Let A_2 (respectively, A_3, A_5, A_7, A_{11}) be the set of multiples of 2^2 (respectively, 3^2, 5^2, 7^2, 11^2) less than 150. The cardinality of the union of these sets is given by

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| + |A_{11}| - |A_2 \cap A_3| - |A_2 \cap A_5| \\
&= \lfloor 150/2^2 \rfloor + \lfloor 150/3^2 \rfloor + \lfloor 150/5^2 \rfloor + \lfloor 150/7^2 \rfloor + \lfloor 150/11^2 \rfloor \\
&\quad - \lfloor 150/(2^2 3^2) \rfloor - \lfloor 150/(2^2 5^2 \rfloor \\
&= 37 + 16 + 6 + 3 + 1 - 4 - 1 \\
&= 58
\end{aligned}$$

Therefore the number of square-free positive integers less than 150 is

$$150 - 58 = 92 \ .$$