Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 26
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Practice Quiz 1

Problem 1 [6 points] **Quantifiers**

Consider the following proposition.

$$\forall x \forall y \ (x < y \rightarrow \exists z \ (x < z \wedge z < y))$$

(a) [2 points] Prove or disprove this proposition, assuming that the universe is the set of integers.

(b) [4 points] Write the negation of this proposition. Your solution may *not* use the $\neg$ symbol, but may use any of the relations $=$, $<$, $>$, $\leq$, or $\geq$.

Problem 2 [10 points] **Induction**

Define the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let M^n denote the product of n copies of the matrix M. Use ordinary induction to prove that for all $n \geq 1$, the upper right entry of M^n is F_n, the n-th Fibonacci number. State your inductive hypothesis clearly.

For reference, the standard formula for the product of 2×2 matrices is given below.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} 1g$$

Recall that $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $\ldots$.

Problem 3 [10 points] **Structural Induction**

Consider the sets $S_1, S_2, \ldots$ defined inductively as

- $S_0 = \{0, 1\}$
- For all $n \geq 1$, $S_n = \left\{\frac{x+y}{2} \mid x, y \in S_{n-1}\right\}$

i.e., where S_n contains the average values of each pair of numbers in S_{n-1}. Note we allow $x = y$ in the definition of S_n.

(a) [1 point] Write out the elements of each of the following sets: S_1, S_2, and S_3.

(b) [2 points] Generalize (in your own words or in symbols) to give the elements of S_n.

(c) [7 points] Prove by induction that for all $n \in \mathbb{N}$, $1/3 \notin S_n$. You will probably want to strengthen the induction hypothesis. You may not assume your generalization from the previous part is true; you must prove that statement if you wish to use it.

Problem 4 [10 points] **Precedence Relations**

Death is planning the end of the world. This involves a number of tasks each of which takes one minute to complete. The prerequisites associated with these tasks are listed below.

Abbrv.	Task	Prerequisites
C	Compose a requiem	
N	Notify the UN	B
D	Signal the daemons	B
B	Blow the trumpet	
T	Sell T-shirts: "All I got was this lousy t-shirt."	N
Q	Grade the 6.042 quiz	S
G	Open the gates	D,N
S	Put out the sun	C,N

(a) [5 points] Represent the tasks and their prerequisites as a directed graph.

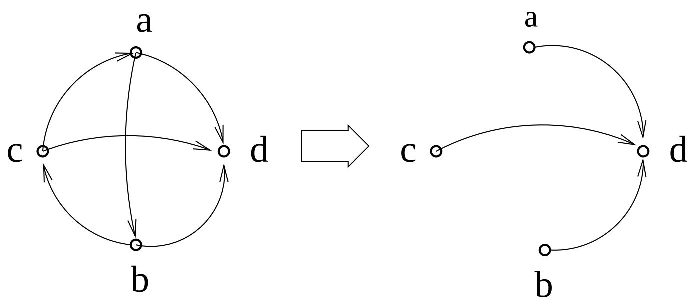
(b) [5 points] Death is omnipotent and can therefore work on as many tasks at a time as he wishes. What is the minimum amount of time required for him to end the world? Why? (An explanation without proof is fine.)

Problem 5 [10 points] **Graphs**

The vertices of a directed graph can be partitioned into *strongly-connected components*. Two vertices u and v belong to the same strongly connected component if there is a path from u to v and a path from v to u. A strongly connected component may consist of a single vertex.

Math Moose has noticed a strange phenomenon. He starts with a directed graph and partitions the vertices into strongly-connected components. He then removes all edges that connect two vertices in the same strongly-connected component. The resulting graph always seems to be acyclic.

An example is worked out in the figure below. The original directed graph is shown on the left. The vertices a, b, and c form one strongly-connected component. The single vertex d forms a second strongly-connected component. Math Moose removes the edges $a \to b$, $b \to c$, and $c \to a$, since these edges connect vertices in the same strongly-connected component. The resulting graph is shown on the right. Note that this graph is acyclic.



Use proof by contradiction to show that for every directed graph, application of Math Moose's procedure gives a directed acyclic graph.

Problem 6 [5 points] **Graphs**

Let $G = (V, E)$ be a simple, undirected graph, and let C be a set of colors. Define a *partial coloring of G using C* to be a function that assigns to each node in V either a color in C or no color. (That is, it colors some, none, or all of the nodes using colors in C.) Define a *partly colored graph* to be a simple, undirected graph $G = (V, E)$ together with a partial coloring of G.

Prove the following fact about partly colored graphs:

> In a partly colored graph, any walk connecting a colored node to an uncolored node must include a colored node adjacent to an uncolored node.

Problem 7 [15 points] **Algorithms**

Let $G = (V, E)$ be a simple, undirected graph. The following algorithm, *RedBlue*, manipulates partial colorings of G using $C = \{\text{red}, \text{blue}\}$. (Partial colorings are defined in the previous problem.)

> Initially, one vertex r of G is colored red, and all the other vertices are uncolored. At each step, one of two events occurs:
>
> a) Some uncolored vertex v that is adjacent to a red vertex becomes blue.
>
> b) Some uncolored vertex v that is adjacent to a blue vertex becomes red.
>
> If neither rule can be applied, the algorithm terminates.

(a) [3 points] Formalize this algorithm as a state machine; that is, define Q, Q_0, and δ.

(b) [3 points] Prove that the algorithm eventually terminates.

(c) [3 points] Using the fact stated in Problem 6, prove that if G is connected, then all vertices are colored when the algorithm terminates.

(d) [3 points] Consider applying this algorithm to *bipartite* graphs, i.e., graphs in which the vertices can be partitioned into two sets A and B such that there is no edge between any pair of vertices in A, and similarly for B. Suppose the node r that is colored red in the initial state is in set A. Using the Invariant Theorem, prove that in any reachable state, only vertices of A are colored red and only vertices of B are colored blue.

(e) [3 points] Combine the previous parts to prove that when run on a connected bipartite graph, the above algorithm terminates and outputs a valid 2-coloring.