6.042 Exam 2

November 15, 2000

Name: ________________________

Tutorial Leader: ________________________

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Problem 1 [25 points] **Summations**

Find the closed form for each of the expressions listed below. Give your answer in simplified form.

(a) **(6 points)**: $\sum_{i=35}^{58} 2i + 1$

(b) **(9 points)**: $\sum_{i=1}^{\infty} \sum_{j=1}^{n} \left(\frac{j^2+2j}{j^2+2j+1}\right)^i$

(c) **(10 points)**: $\sum_{i=0}^{n} 2 \cdot 3^i \cdot \binom{n}{i}$

Problem 2 [10 points] **Bounding Infinite Sums**

Using the Integral Method, give a tight upper-bound for

$$\sum_{i=1}^{n} \frac{3}{i^{10}}$$

Problem 3 [30 points] **Asymptotic Notation**

Determine which of the eight choices below best describes each function's asymptotic behavior. No proofs are necessary. **(5 points each)**

$\theta(n)$	$\theta(n!)$
$\theta(n \log n)$	$\theta(n^2)$
$\theta(1)$	$\theta(2^n)$
$2^{\theta(n)}$	None of the above

(a) $f(n) = \frac{n^2+2}{4+\frac{1}{n}}$.

(b) $g(n) = n + \ln n + (\ln n)^2$.

(c) $h(n) = \frac{n^2+2n-3}{n^2-1}$

(d) $j(n) = \sum_{i=0}^{n} 2^{2i+1}$

(e) $k(n) = (2 + \sin n)2^{n+\sin(n)}$.

(f) $l(n) = \ln((n^2)!)$.

Problem 4 [50 points] **3-coloring Grid Graphs**

In Exam 1, we proved that an m by n grid graph is always 2-colorable. Thus, it must be 3-colorable. However, with 3 colors, we suddenly get a lot of choices for how to color the graph. In this problem we will solve part of the question "How many ways are there to 3-color an m by n grid graph?"

The general case is too difficult for an exam question. However, we can come up with solutions to some more restricted questions.

(Recall that a 3-coloring of a graph is an assignment of one of three colors to each vertex in the graph such that if two vertices are connected by an edge, they must have different colors. We consider two 3-colorings different if any vertex is colored differently between the two.)

(a) (15 points) How many ways are there to 3-color a 2 by n grid graph? Give your answer in closed form. (Hint: come up with a recurrence relation for C_n, the number of ways to color a 2 by n grid graph, in terms of C_{n-1}, and figure out C_1 directly.)

(b) (8 points) Now, we will approach the problem of how to 3-color a 3 by n grid graph. Suppose we want to color the graph by first coloring the first $n-1$ columns, then choosing a coloring for the last column. However, the number of ways we can color the last column is dependent on which coloring we choose. There are two cases to consider.

In the first case, the vertices $(n-1,0)$ and $(n-1,2)$ are the same color.

In the second case, the vertices $(n-1,0)$, $(n-1,1)$, and $(n-1,2)$ are all different colors.

1. How many ways can we color the last column in the first case?
2. Of the ways to color the last column in the first case, how many use three different colors?
3. How many ways can we color the last column in the second case?
4. Of the ways to color the last column in the second case, how many use three different colors?

(c) (7 points) Let A_n be the number of ways to 3-color a 3 by n grid graph such that the last column is colored with only 2 colors. Let B_n be the number of ways to 3-color a 3 by n grid graph such that the last column is colored with three different colors.

Formulate a system of recurrence relations for A_n and B_n.

(d) (4 points) Find the values A_1 and B_1.

(e) **(10 points)** Find c_1, c_2, d_1, and d_2 such that

$$A_n = xc_1^n + yc_2^n$$
$$B_n = xd_1c_1^n + yd_2c_2^n$$

is the general form of the solution to this system of recurrence relations.

(f) **(6 points)** Find a closed-form expression for the number of ways to 3-color a 3 by n grid graph.

Problem 5 [20 points] **Rearranging the Alphabet**

For this problem you do not need to evaluate an expression fully: you may leave it in terms of binomial coefficients, powers of numbers, factorials, and so on.

(a) (10 points) How many ways can you rearrange the alphabet so that the words "TEST," "EXAM," and "STUDY" do not appear as consecutive letters in your arrangement?

(b) (10 points) How many ways are there to choose a collection of 26 letters (not necessarily distinct) such that the words "EXAM," "STUDY," and "TEST" cannot be formed using the letters in the collection?

Problem 6 [14 points] **A Strange Set**

Define the function $fp(x) = x - \lfloor x \rfloor$. ($fp$ stands for "fractional part.")

Let $S = \{n \in \mathbb{Z} \mid n \neq 0 \wedge fp(n\sqrt{2}) < 0.01\}$.

Use the pigeonhole principle to prove that S is nonempty.

Problem 7 [12 points] **Winning Record**

Suppose that a tennis player's record this year is 31 wins and 5 losses. Answer each of the following true/false. No explanation is needed.

(a) **(3 points)** The player must have won 5 games in a row.

(b) **(3 points)** The player must have won 6 games in a row.

(c) **(3 points)** If the player's record was 30 wins and 5 losses, the player must still have won 5 games in a row.

(d) **(3 points)** If the player's record was 30 wins and 5 losses, the player must still have won 6 games in a row.

Problem 8 [24 points] **Counting Binary Strings**

Let $s = s_1 s_2 \ldots s_{2n}$ be a string of 0s and 1s. The *reverse* of s is the string $R(s) = s_{2n} s_{2n-1} \ldots s_1$.

(a) **(8 points)** A string that is equal to its reverse is called a *palindrome*. How many palindromes of length $2n$ are there?

(b) **(8 points)** How many strings of length $2n$ have exactly $2r$ bits in common with their reverse?

(c) **(8 points)** A string s is *1-derful* if all the common bits of s and $R(s)$ are 1 bits. How many 1-derful strings are there of length $2n$ which have $2r$ bits in common with their reverse?

Problem 9 [15 points] **Finite and Infinite Sets**

For each of the following sets, state whether it is finite, countably infinite, or uncountably infinite. If it is finite, give its size. If it is countably infinite, give an injective function from the given set to $\mathbb{N}$. If it is uncountable, give a brief argument.

(a) **(5 points)** $\mathcal{P}(\mathbb{N}) \cap \mathcal{P}(\{2^{-i} \mid i \in \mathbb{N}\})$

(b) **(5 points)** N, the set of all finite sets of prime numbers.

(c) **(5 points)** C, the set of all equivalence classes of the equivalence relation $\{(x, y) \mid x, y \in \mathbb{R} \text{ and } x - y \in \mathbb{Q}\}$