Series and Summations

Examples:

Annuities

Book Stacking

Important for algorithms and systems

We would like to have *closed form*

At least approximate its value well enough

Would you prefer a million dollars today or \$50,000 a year for the rest of your life?

An n-year, m-payment annuity pays m dollars at the start of each year for n years.

We estimate the fixed annual interest rate p around 8%

Ten dollars will be
$(1+p) = \$10.80$ in a year,
$(1+p)^2 \approx \$11.66$ in two years,
and so forth.

Looked at another way, ten dollars paid out a year from now are only really worth $1/(1+p) \approx$ \$9.26 today.

The reason is that if we had the \$9.26 today, we could invest it and would have \$10.00 in a year anyway.

Therefore, p determines the value of money paid out in the future.

First payment of m dollars is truly worth m dollars.

Second payment a year later is worth only $m/(1+p)$,

Third payment is worth $m/(1+p)^2$,

n-th payment is only $m/(1+p)^{n-1}$,

The total value V is:

$$V = \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}}$$

"Closed form" mathematical expression without any summation or product.

Summation of a *geometric series* is:

Theorem: For all $n \geq 1$ and all $x \neq 0$, $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$.

We have:

$$\begin{aligned} V &= \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} \\ &= \sum_{j=0}^{n-1} \frac{m}{(1+p)^{j}} \quad \text{(substitute } j = i-1\text{)} \\ &= m \sum_{j=0}^{n-1} x^j \quad \text{(substitute } x = \frac{1}{1+p}\text{)} \end{aligned}$$

Thus we have $\frac{m-mx^n}{1-x}$ where x is $\frac{1}{1+p}$.

So for $m =$\$50,000, $n =$20, and $p = 0.8$ we have $V \approx \$530,180$. Better have \$550,000 now!

So we need to prove the **Theorem:**
$\forall n \geq 1$ and all $\forall x \neq 0$, $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$.

We prove by induction on n.
$P(n)$ $\forall x \neq 1$ $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$.

Base: $P(1)$ holds, since
$\sum_{i=0}^{0} x^i = x^0 = 1$ and $\frac{1-x^1}{1-x} = 1$.

Step:
Assume $\forall x \neq 1$, $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$, and prove $\forall x \neq 1$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$.

$$
\begin{aligned}
\sum_{i=0}^{n} x^i &= x^n + \sum_{i=0}^{n-1} x^i \\
&= x^n + \frac{1 - x^n}{1 - x} \\
&= \frac{x^n(1 - x) + 1 - x^n}{1 - x} \\
&= \frac{1 - x^{n+1}}{1 - x}
\end{aligned}
$$

The Sum of a Geometric Series L11–P. 8

How the formula was found??

The trick is to let S be the value of the sum:

$$\begin{aligned} S &= 1 + x + x^2 + \ldots + x^{n-1} \\ -xS &= -x - x^2 - x^3 - \ldots - x^n \end{aligned}$$

Adding these two equations gives:

$$\begin{aligned} (1-x)S &= 1 - x^n \\ S &= \frac{1-x^n}{1-x} \end{aligned}$$

Assume you will live forever ...
(always in a good shape of course)

Would you prefer a million dollars today or \$50,000 a year? (Yes! infinite money)

The limit of our geometric sum as n tends to infinity is

Theorem If $|x| < 1$, then $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$.

$$\begin{aligned} \sum_{i=0}^{\infty} x^i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} x^i \\ &= \lim_{n\to\infty} \frac{1-x^n}{1-x} \\ &= \frac{1}{1-x} \end{aligned}$$

Note that in evaluating the limit the x^n term vanishes
since we assumed that $|x| < 1$.

$x = 1/(1+p) < 1$, so the theorem applies.
Substituting for x, we get an annuity value of

$$\begin{aligned} V &= m \cdot \frac{1}{1-x} \\ &= m \cdot \frac{1}{1 - 1/(1+p)} \\ &= m \cdot \frac{1+p}{(1+p)-1} \\ &= m \cdot \frac{1+p}{p} \end{aligned}$$

Plugging in $m = \$50,000$ and $p = 0.8$ gives only $V = \$675,000$.
Amazingly, a million dollars today is worth much more than $\$50,000$ paid every year forever!

Because with a million dollars today in the bank earning 8% interest,
We could spend $\$80,000$ a year forever.

$$
\begin{aligned}
1 + 1/2 + 1/4 + 1/8 + \ldots &= \sum_{i=0}^{\infty} (1/2)^i \\
&= \frac{1}{1 - (1/2)} \\
&= 2 \\
0.999999999\ldots &= 0.9 \sum_{i=0}^{\infty} (1/10)^i \\
&= 0.9 \frac{1}{1 - 1/10} \\
&= 0.9 \frac{10}{9} \\
&= 1 \\
1 - 1/2 + 1/4 - 1/8 + \ldots &= \sum_{i=0}^{\infty} (-1/2)^i \\
&= \frac{1}{1 - (-1/2)} \\
&= 2/3
\end{aligned}
$$

Sums of Increasing Geometric Series

$$
\begin{aligned}
1 + 2 + 4 + 8 + \ldots + 2^{n-1} &= \sum_{i=0}^{n-1} 2^i \\
&= \frac{1 - 2^n}{1 - 2} \\
&= 2^n - 1 \\
1 + 3 + 9 + 27 + \ldots + 3^{n-1} &= \sum_{i=0}^{n-1} 3^i \\
&= \frac{1 - 3^n}{1 - 3} \\
&= \frac{3^n - 1}{2}
\end{aligned}
$$

Rule of thumb: approximately equal to the greatest term value there are cases in which we need to work harder...

$\sum_{i=1}^{n} ix^i = x + 2x^2 + 3x^3 + \ldots + nx^n$

$\frac{d}{dx} \sum_{i=0}^{n} x^i = \frac{d}{dx} \frac{1-x^{n+1}}{1-x}$

$$\begin{aligned}\sum_{i=0}^{n} ix^{i-1} &= \frac{-(n+1)x^n(1-x)-(-1)(1-x^{n+1})}{(1-x)^2} \\ &= \frac{-(n+1)x^n+(n+1)x^{n+1}+1-x^{n+1}}{(1-x)^2} \\ &= \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}\end{aligned}$$

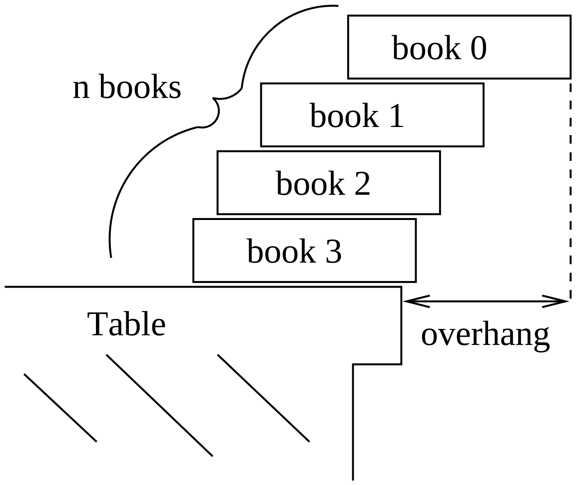
Now we multiply be x to have:

$\sum_{i=0}^{n} ix^i = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$

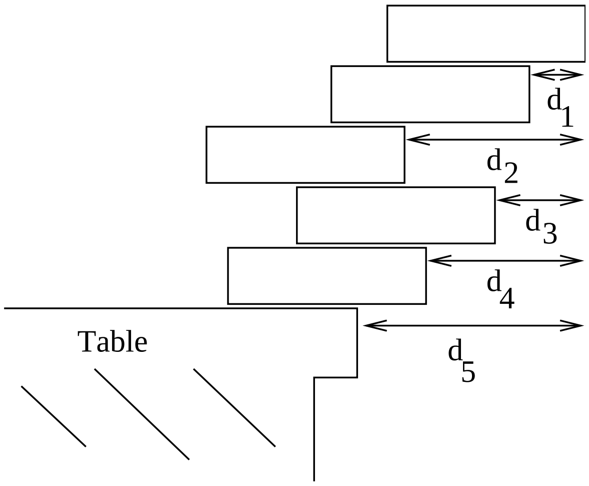
Good practice to verify by induction, at least examples...

In case $n = \infty$, we may conclude:

Theorem: If $|x| < 1$, then $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$.

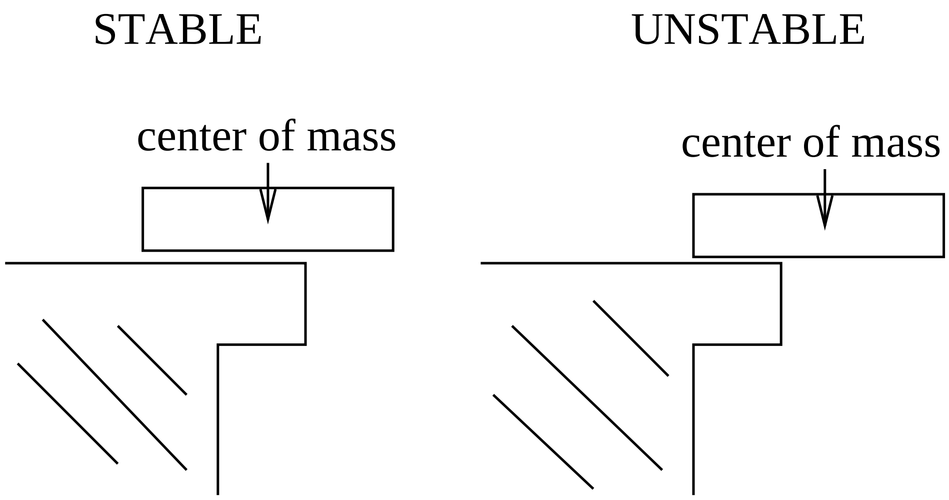


If n books are stacked on a table,
how far can the top book overhang the edge
without the whole stack falling over?



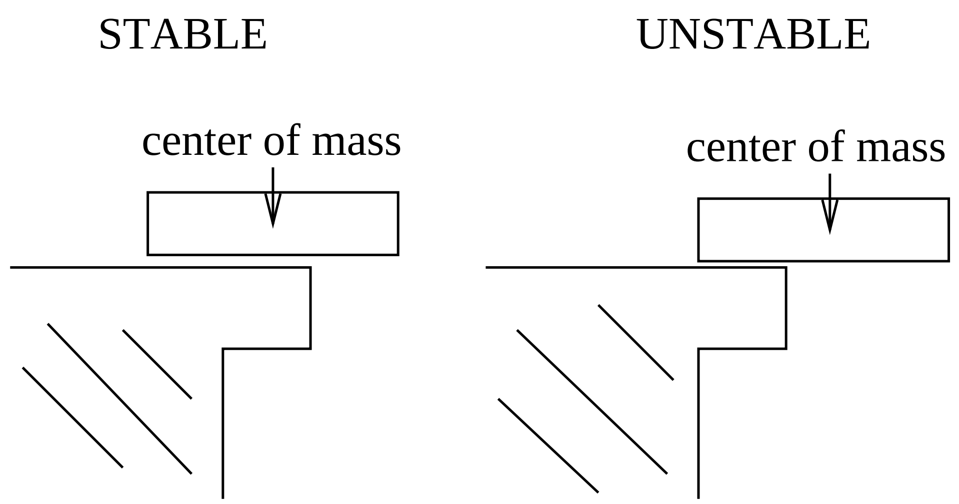
Book length = 2, book mass identical.
$1 \leq i \leq n$, d_i is the distance that book 0 extends beyond the right end of book i.
The table is regarded as book $n + 1$

A stack of books is *stable* iff for $1 \leq k \leq n$, the center of mass for the top k books lies above the $(k+1)$-st book.



Greedy strategy:
The book 1 as far as possible on the book 2
The two top books as far as possible on book 3
And so on...

d_k is the center of the mass of the top k books



The center of mass of book i is at distance d_i+1 to the left of the top book.

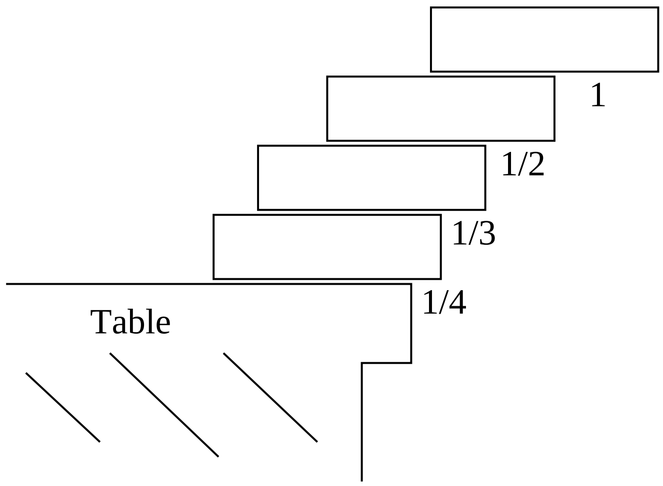
The center of mass of the top k books $(0,\ldots,k-1)$ is the average of this values:

$\frac{(d_{k-1}+1)+(d_{k-2}+1)+\cdots+(d_0+1)}{k}$

By our greedy choice:
$d_k = \frac{(d_{k-1}+1)+(d_{k-2}+1)+\cdots+(d_0+1)}{k}$

Or: $kd_k = d_{k-1} + d_{k-2} + \cdots + d_0 + k$

Now algebra:
$(k-1)d_{k-1} = d_{k-2} + d_{k-3} + \cdots + d_0 + k - 1$
Substraction of the above formulas:
$kd_k - (k-1)d_{k-1} = d_{k-1} + 1$
or, $d_k = d_{k-1} + 1/k$
$d_k = d_{k-2} + 1/(k-1) + 1/k$
...
$d_k = 1 + 1/2 + \cdots + 1/(k-1) + 1/k$



We cannot do better since we mast have the centers located such that

$$d_k \le \frac{(d_{k-1}+1)+(d_{k-2}+1)+\cdots+(d_0+1)}{k}$$

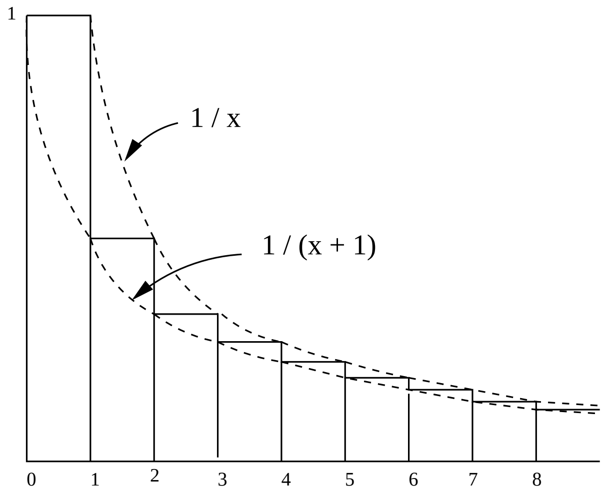
using induction we can prove that d_k is largest possible.

n-th Harmonic number H_n is:
$\sum_{i=1}^{n} \frac{1}{i}$

$H_1 = 1$, $H_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{16}$, $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

$H_4 > 2$ thus — the total extension of a 4-book stack is greater than one full book!

What about extension of 100 books?
Look for $H_l \geq 200$,
Can we have a "closed form"?
No close form is known



The area under the "stairstep" curve over the interval $[0, n]$ is equal to $H_n = \sum_{i=1}^{n} 1/i$.

$1/x \geq$ the stairstep function
$1/(x+1) \leq$ the stairstep function

$H_n \leq 1 + \int_1^n \frac{1}{x}\, dx = 1 + \ln n$
$H_n \geq \int_0^n \frac{1}{x+1}\, dx = \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1)$

We need approximately $e^2 00$ books, which is a 86 digits number!.

Better approximation —

$H_n = \ln n + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4}$

$\gamma = 0.577215664\ldots$ called Euler's constant

$\epsilon(n)$ is between 0 and 1 for all n.

Shorthand $H_n \sim \ln n$ indicates that the leading term of H_n is $\ln n$.

$f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \to 1$.

$\sim$ notation is useful when we do not care about lower order terms.

For example, if $n = 100$, then we can compute $H(n)$ to great precision using only the two leading terms:

$|H_n - \ln n - \gamma| \leq |\frac{1}{200} - \frac{1}{120000} + \frac{1}{120 \cdot 100^4}| < \frac{1}{200}$