Staff Solutions to Problem Set 7

Reading: Sections 12.9

Problem 1.
Prove Corollary 12.9.12: If all edges in a finite weighted graph have distinct weights, then the graph has a unique MST.

Hint: Suppose $M$ and $N$ were different MST’s of the same graph. Let $e$ be the smallest edge in one and not the other, say $e \in M - N$, and observe that $N + e$ must have a cycle.

Solution. Assume for the sake of contradiction that $M$ and $N$ were different MST’s of the same graph. Let $e$ be a minimum weight edge as in the hint.

Since $N$ is a spanning tree, we know that $N + e$ is connected, and it has too many edges to be a tree, so $N + e$ has a cycle. Since $M$ has no cycles, the cycle in $N + e$ cannot consist solely of edges from $M$. So there must be an edge $g \in N - M$ on the cycle, and we know that $w(g)$ must be larger than $w(e)$ by definition of $e$. Removing $g$ from $N + e$ leaves a connected graph with the same number of nodes and edges as $N$, so $N + e - g$ must be a spanning tree. But $N + e - g$ weighs $w(g) - w(e)$ less than $N$, contradicting the fact that $N$ is a minimum weight spanning tree.

Problem 2.
This problem generalizes the result proved Theorem 12.6.3 that any graph with maximum degree at most $w$ is $(w+1)$-colorable.

A simple graph, $G$, is said to have width $w$ iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most $w$ vertices that precede it in the sequence. If the degree of every vertex is at most $w$, then the graph obviously has width at most $w$—just list the vertices in any order.

(a) Prove that every graph with width at most $w$ is $(w+1)$-colorable.

Solution. We use induction on $n$, the number of vertices. Let $P(n)$ be the proposition that for all $w$, every $n$-vertex graph with width $w$ is $(w+1)$-colorable.

Base case: ($n = 1$) Every graph with 1 vertex has width 0 and is $0+1 = 1$ colorable. Therefore, $P(1)$ is true.

Inductive step: Now we assume $P(n)$ in order to prove $P(n+1)$. Let $G$ be an $(n+1)$-vertex graph with width at most $w$. This means that the $n+1$ vertices can be arranged in a sequence, $S$, such that each vertex is connected to at most $w$ preceding vertices. Removing the last vertex, $v$, and all edges incident to it gives a subgraph $G'$ with $n$ vertices. The subgraph $G'$ also has width at most $w$, since the sequence $S$ with its last vertex removed is a sequence of all the vertices of $G'$ with each vertex adjacent to exactly the same previous vertices. So by Induction Hypothesis, $G'$ is $(w+1)$-colorable. But any $(w+1)$-coloring of $G'$ can be extended to a $(w+1)$-coloring of $G$ by assigning a color to $v$ that differs from the colors of its adjacent vertices. Since there are at most $w$ colors among the $w$ vertices adjacent to $v$, there will always be a different
one of the $w + 1$ colors to assign to $v$. So $G$ is $(w + 1)$-colorable, which proves $P(n + 1)$. This completes the proof of the Induction step.

The result now follows for all $G$ by the Principle of Induction.

(b) Describe a 2-colorable graph with minimum width $n$.

**STAFF NOTE:** *Hint: The complete bipartite graph $K_{n,n}$.***

**Solution.** An example would be the complete bipartite graph $K_{n,n}$. Since it is bipartite, it is 2-colorable. But every vertex has degree $n$ and therefore has minimum width $n$, since the last vertex in any list of the vertices is adjacent to $n$ preceding vertices.

(c) Prove that the average degree of a graph of width $w$ is at most $2w$.

**Solution.** If we line up the vertices, we can define the backdegree of a vertex to be the number of preceding vertices it is adjacent to. The sum of the back degrees equals the number, $e$, of edges. Since there is a sequence in which all the back degrees are at most $w$, the total number of edges is at most $w$ times the number, $n$, of vertices. But by the Handshaking Lemma, the sum of all the degrees is $2e$, so the average degree is $2e/n \leq 2wn/n = 2w$.

(d) Describe an example of a graph with 100 vertices, width 3, but average degree more than 5.

*Hint: Don’t get stuck on this; if you don’t see it after five minutes, ask for a hint.*

**Solution.** The hint is to line up the 100 vertices and have each vertex be adjacent to the 3 immediately preceding vertices, if any. By definition of width, this graph has width 3. All vertices other than the first three are now adjacent to three preceding vertices, and all vertices except the last three are also adjacent to the three following vertices. So vertices 4 through 97 all have degree 6; this alone ensures that the average degree is at least $6 \cdot 94/100 = 5.76$.

Problem 3.

A basic example of a simple graph with chromatic number $n$ is the complete graph on $n$ vertices, that is $\chi(K_n) = n$. This implies that any graph with $K_n$ as a subgraph must have chromatic number at least $n$. It’s a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no triangle—length three cycle—and hence no subgraph isomorphic to $K_n$ for $n \geq 3$. Namely, let $G$ be the 11-vertex graph of Figure 2. The reader can verify that $G$ is triangle-free.

(a) Show that $G$ is 4-colorable.

**Solution.** Figure 1 shows a valid coloring.

(b) Prove that $G$ can’t be colored with 3 colors.

**Solution.** Assume by contradiction that there is one coloring using only 3 colors: red, blue and green. The outer pentagon of the graph is an odd-length cycle, and so requires all 3 colors. So we can assume wlog that the outer pentagon is colored as shown in the left hand side of Figure 3.

This coloring of the pentagon forces the coloring of three interior points, as shown in the right hand side of Figure 3. Now the point in the center has neighbors with all three colors, so it is impossible to color it.
Figure 1  Coloring using 4 colors

The graph $G$ is the known as the Groetzsch graph. It is the smallest triangle-free graph with chromatic number 4. It turns out that for any $n > 0$, there is a triangle-free graph with chromatic number $n$, see Mycielski graphs on the Wolfram Mathworld web site.

Problem 4.
Take a regular deck of 52 cards. Each card has a suit and a value. The suit is one of four possibilities: heart, diamond, club, spade. The value is one of 13 possibilities, $A, 2, 3, \ldots, 10, J, Q, K$. There is exactly one card for each of the $4 \times 13$ possible combinations of suit and value.

Ask your friend to lay the cards out into a grid with 4 rows and 13 columns. They can fill the cards in any way they’d like. In this problem you will show that you can always pick out 13 cards, one from each column of the grid, so that you wind up with cards of all 13 possible values.

(a) Explain how to model this trick as a bipartite matching problem between the 13 column vertices and the 13 value vertices. Is the graph necessarily degree-constrained?

Solution. Create a simple bipartite graph with 13 column vertices and 13 value vertices. Connect a column to a value by a single edge iff a card with that value is contained in that column. A perfect matching would then indicate the value of the card you would choose from each column.

The graph may not be degree-constrained if any one of the columns contains more than one card with the same value. In the case where the matching indicates a value that appears more than once in the column it is matched to, you can arbitrarily pick any card of that value in that column.

(b) Show that any $n$ columns must contain at least $n$ different values and prove that a matching must exist.

Figure 2  Graph $G$ with no triangles and $\chi(G) = 4$. 
Solution. If \( S \) is a set of columns, they contain \( 4|S| \) cards. No card value repeats more than four times, so at least \( |S| \) values must appear among those cards. Thus \( |N(S)| \geq |S| \) and Hall’s theorem gives us a matching.

Problem 5.
Let \( G \) be a digraph. The neighbors of a vertex \( v \) are the endpoints of the edges out of \( v \). Since a digraph is formally the same as a binary relation on \( V(G) \), the set of neighbors of \( v \) is simply the image, \( G(v) \), of \( v \) under the relation \( G \).

(a) Suppose \( f \) is an isomorphism from \( G \) to another digraph \( H \). Prove that

\[
f(G(v)) = H(f(v)).
\]

Your proof should follow by simple reasoning using the definitions of isomorphism and image of a vertex under the edge relation—no pictures or handwaving.

Hint: Prove by a chain of iff’s that

\[
h \in H(f(v)) \iff h \in f(G(v))
\]

for every \( h \in V(H) \).

STAFF NOTE: Hint: Use the fact that \( h = f(u) \) for some \( u \in V(G) \).

Solution. Proof. Suppose \( h \in V(H) \). By definition of isomorphism, there is a unique \( u \in V(G) \) such that \( f(u) = h \). Then

\[
\begin{align*}
h \in H(f(v)) & \iff \langle f(v) \rightarrow h \rangle \in E(H) & \text{(def of } H(f(v)) \text{)} \\
& \iff \langle f(v) \rightarrow f(u) \rangle \in E(H) & \text{(def of } u \text{)} \\
& \iff \langle v \rightarrow u \rangle \in E(G) & \text{(since } f \text{ is an isomorphism)} \\
& \iff u \in G(v) & \text{(def of } G(v) \text{)} \\
& \iff f(u) \in f(G(v)) & \text{(def of } f \text{-image)} \\
& \iff h \in f(G(v)) & \text{(def of } u \text{)}
\end{align*}
\]

So \( H(f(v)) \) and \( f(G(v)) \) have the same members and therefore are equal.

(b) Conclude that if \( G \) and \( H \) are isomorphic graphs, then they have the same number of vertices of out-degree \( k \), for all \( k \in \mathbb{N} \).
**STAFF NOTE:** *Hint: outdeg(v) ::= |G(v)|.*

**Solution.** Since an isomorphism is a bijection, any set of vertices and its image under an isomorphism will be the same size (Bijection Mapping Rule 4.7). So part (a) implies that an isomorphism, $f$, maps out-degree $k$ vertices to out-degree $k$ vertices. This means that the image under $f$ of the set of out-degree $k$ vertices of $G$ is precisely the set of out-degree $k$ vertices of $H$. So by the Mapping Rule again, there are the same number of out-degree $k$ vertices in $G$ and $H$. 