Staff Solutions to 2nd Conflict Midterm Exam November 5

STAFF NOTE: Topics: RSA, Digraphs, DAGs, Partial orders and equivalence, Simple Graphs, Connectivity and Trees, Sums and Products, and Asymptotics

Problem 1 (Graphs, DAGs, and Partial Orders) (10 points).
A tennis tournament consists of a series of matches between two players. Usually the objective is to determine a single best player among a set of players. The organizers of the Math for Computer Science tournament want to do more: they want to find a linear ranking of all the players. To avoid controversy, they want to avoid the awkward situation of having a sequence of players each of whom beats the next player in the sequence and then having last player beat the first. So the organizers will keep a running record of who beat whom during the tournament and never allow simultaneous matches whose outcomes could lead to an awkward situation.

Knowledge of binary relations can help the organizers in arranging the tournament. Namely, at any stage of the tournament, the organizers have a record of who lost to whom. Mathematically, we can say that there is a binary relation, $L$, on players where $p L q$ means that player $p$ lost a match to player $q$. No awkward situations means that the positive length walk relation, $L^+$, is a strict partial order. Indicate which of the following partial order concepts correspond to the properties (a)–(j) of the partial order $L^+$.

Partial Order Concepts

- comparable
- incomparable
- maximum
- maximal
- minimum
- minimal
- a chain
- an antichain
- reflexive
- irreflexive
- asymmetric
- a topological sort
- a linear order

(a) An unbeaten player so far is a ____________ element.

Solution. maximal

(b) A player who has lost every match he was in is a ____________ element.

Solution. minimal

(c) A player who is sure to rank first at the end of the tournament is a ____________ element.

Solution. maximum

(d) A set of players whose rankings relative to each other are unique is ____________.

Solution. a chain
(e) Two players can be matched in the next stage of the tournament only if they are incomparable elements.

Solution. incomparable

(f) The final ranking at the end of the tournament will be topological sort.

Solution. a topological sort

(g) No more matches are possible if and only if $L^+$ is a linear order.

Solution. a linear order

(h) A set of players any two of whom could be paired up to play the next match is antichain.

Solution. an antichain

(i) The fact that no player loses to himself corresponds to $L^+$ being irreflexive.

Solution. irreflexive

(j) If there are 256 players, what is the smallest number of matches that could possibly have been played in a completed tournament?

Solution. 255

Problem 2 (Coloring) (14 points).
Next year, Math for Computer Science will be taught using recitations. Six hour long recitations, each taught by a team of two or three TAs, will be needed. The planned assignment of TAs to recitation sections is as follows:

- R1: Elsa, Anna, Kristoff
- R2: Anna, Kristoff
- R3: Kristoff, Hans, Sven
- R4: Hans, Sven
- R5: Sven, Olaf, Elsa
- R6: Olaf, Elsa

Two recitations cannot be held in the same one-hour time slot if some TA is assigned to teach both recitations. The Registrar must determine the minimum number of time slots required to complete all the recitations.
(a) Recast the Registrar’s problem of determining the minimum number of time slots as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

**Solution.** Each vertex in the graph below represents a recitation section. An edge connects two vertices if the corresponding recitation sections share a staff member and thus can not be scheduled at the same time. The color of a vertex indicates the time slot of the corresponding recitation.

![Graph](image)

(b) Give a coloring of this graph using the fewest possible colors, and explain why no fewer colors are possible. Describe a possible recitation schedule using a minimum number of hour long time slots based on this coloring.

**Solution.** Three colors are necessary and sufficient. Three are sufficient as shown by the coloring:

- red: R1, R4
- white: R2, R5
- blue: R3, R6

This corresponds having three recitation times, where recitations assigned the same color run at the same time. Other schedules are also possible.

Three colors are necessary because each of the three vertices R1, R2, and R3 are adjacent to other two, so they must all have different colors.

Another way to say this is that the subgraph with these vertices is isomorphic to $K_3$, which we know requires three colors.

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**Problem 3 (Simple Graphs) (20 points).**

Starting with some simple graph $G$ with two or more vertices, keep applying the following operations: pick two vertices $u \neq v$ such that either

(i) there is an edge between $u$ and $v$, and there is also a path from $u$ to $v$ which does not include this edge; in this case, delete the edge $(u-v)$.

(ii) there is no path from $u$ to $v$; in this case, add the edge $(u-v)$.

Continue until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.
(a) Let $c$ be the number of connected components and $e$ the number of edges of a simple graph. For each of the following derived variables argue whether it is weakly decreasing, strictly decreasing, or none of the two: $u = e$, $v = c + e$, and $w = 2c + e$.

Solution. Operation (i) decreases $e$ by one and leaves $c$ unchanged since $u$ and $v$ remain connected. So, all three derived variables decrease by one.

Operation (ii) decreases $c$ by one since the two connected components it connects become a single component. It increases $e$ by one. So $u$ increases by one, $v$ stays the same, and $w$ changes by $(2 \cdot -1) + 1 = -1$, that is, it decreases by one.

Therefore, $v$ is weakly decreasing, $w$ is strictly decreasing, and none of the two holds for $u$.

(b) Prove that if the procedure terminates, the remaining graph is a tree.

Solution. We use the characterization of a tree as a cycle-free, connected, simple graph. A final graph must be connected, because otherwise there would be two vertices with no path between them, and then a transition adding the edge between them would be possible. A final graph can’t have a cycle, because deleting any edge on the cycle would be a possible operation.

Problem 4 (Simple Graphs) (20 points). (a) Give an example of a connected, weighted simple graph with four vertices and distinct edge weights that contains a cycle and in which the edge with the largest weight belongs to the minimum weight spanning tree (MST) of the graph.

Solution. $V = \{a, b, c, d\}$, $E = \{a\rightarrow b, b\rightarrow c, b\rightarrow d, c\rightarrow d\}$, $W = \{4, 3, 2, 1\}$.

(b) Can you construct such an example with three vertices? Explain.

Solution. No. A cycle must contain at least three vertices. Since there are only three vertices, they must be all connected in order to form a cycle. It follows that the edge with the largest weight must also be on that cycle. But then it is not possible for the largest edge to be in MST, since taking the other two edges would result in an MST with smaller weight.

(c) Let $G$ be a connected, weighted simple graph. Let $v \in V(G)$ be an arbitrary vertex whose incident edges have distinct weights. Suppose that edge $\langle v\rightarrow w \rangle \in E(G)$ has minimum weight among the edges incident to $v$. Prove that $\langle v\rightarrow w \rangle$ must be an edge of some minimum weight spanning tree (MST) of $G$.

Hint: Appeal to the gray edge construction in the text. Alternatively, consider a path in an MST from between $v$ and $w$.

Solution. Using the gray edge Lemma, color vertex $v$ black and all other vertices white. Then the only gray edges are the edges incident to $v$, so $\langle v\rightarrow w \rangle$ will be the minimum weight gray edge, and therefore will be in the MST built starting with this coloring. Alternatively, if $\langle v\rightarrow w \rangle$ is not in some MST, $M$, then there is a path in $M$ from $v$ to $w$, and this path must start with some edge $g$ incident to $v$ that is heavier than $\langle v\rightarrow w \rangle$. Then $M - g + \langle v\rightarrow w \rangle$ would be an MST with smaller weight than $M$, a contradiction. So $\langle v\rightarrow w \rangle$ must be a member of every MST for $G$.

STAFF NOTE: variation of ps6 question

\(^1\)Actually, the minimum weight edge incident to $v$ must be in every MST, but you need not prove this.
Problem 5 (Simple graphs) (16 points).
Let \( f, g \) be positive real-valued functions on finite trees with at least two vertices. We will extend the \( O() \) notation to such tree functions as follows: \( f = O(g) \) iff there is a constant \( c > 0 \) such that
\[
f(T) \leq c \cdot g(T) \text{ for all trees } T.
\]
For each of the following assertions, state whether it is True or False and briefly explain your answer. You are not expected to offer a careful proof or detailed counterexample.

\( \text{Reminder: } V(T) \text{ is the set of vertices and } E(T) \text{ is the set of edges of } T. \)

(a) \( |V(T)| = O(|E(T)|) \).

**Solution. True.**
\[
|V(T)| = |E(T)| + 1 \leq 2 \cdot |E(T)|.
\]

(b) \( |E(T)| = O(|V(T)|) \).

**Solution. True.**
\[
|E(T)| = |V(T)| - 1 \leq |V(T)|.
\]

(c) \( |V(T)| = O(\chi(T)) \), where \( \chi(T) \) is the chromatic number of \( T \).

**Solution. False.**
For example, every tree with more than one vertex has a chromatic number 2.

(d) \( \chi(T) = O(|V(T)|) \).

**Solution. True.**
Assigning a different color to each vertex gives a valid coloring, so \( \chi(T) \leq |V(T)| \).

Problem 6 (Sums) (10 points).
We are interested in finding a closed form formula for the sum
\[
\sum_{i=1}^{n} (2i - 1)(i + 1).
\]
Display such a closed form formula for (1). Alternatively, you can get full credit by giving a clear description of a procedure for deriving such a closed formula without fully executing your procedure to derive an explicit closed formula.

**Solution.** As in Section 14.2, we can (correctly) guess that a formula for (1) will be a cubic polynomial
\[
an^3 + bn^2 + cn + d.
\]
Then we can determine the parameters \( a, b, c, \) and \( d \) by plugging in a few values for \( n \) until we get enough equations in \( a, b, c, d \) to solve for their values. Applying this method to our example gives:

\[
\begin{align*}
  n = 1 & \implies 2 = a + b + c + d \\
  n = 2 & \implies 11 = 8a + 4b + 2c + d \\
  n = 3 & \implies 31 = 27a + 9b + 3c + d \\
  n = 4 & \implies 66 = 64a + 16b + 4c + d.
\end{align*}
\]

Solving this system gives the solution \( a = 2/3, b = 3/2, c = -1/6, d = 0 \) So the closed form formula for (1) would be

\[
\frac{4n^3 + 9n^2 - n}{6}. \tag{2}
\]

Of course this remains a guess until we have verified it somehow, for example, by induction.

Another approach is to rewrite \((2i - 1)(i + 1)\) as \(2i^2 + i - 1\) and then use the known formula for the sum of squares:

\[
\sum_{i=0}^{n} i^2 = \frac{(2n + 1)(n + 1)n}{6}
\]

So,

\[
\sum_{i=1}^{n} (2i - 1)(i + 1) = \sum_{i=1}^{n} (2i^2 + i - 1)
\]

\[
= 2 \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1
\]

\[
= 2 \left( \frac{(2n + 1)(n + 1)n}{6} \right) + \frac{n(n + 1)}{2} - n
\]

\[
= \frac{2}{6} (2n^3 + 3n^2 + n) + \frac{1}{6} (3n^2 + 3n - 6n)
\]

\[
= \frac{1}{6} (4n^3 + 9n^2 - n).
\]

which agrees with (2).

A third approach involves using generating functions as described in Chapter 16, but let’s not get ahead of ourselves.