Staff Solutions to In-Class Problems Week 2, Mon.

Problem 1.
Say a number of cents is makeable if it is the value of some set of 6 cent and 15 cent stamps. Use the Well Ordering Principle to show that every integer that is a multiple of 3 and greater than or equal to twelve is makeable.

Solution. Proof. Let

\[ C := \{ n \geq 12 \mid n \text{ is a multiple of 3 and is not makeable} \} \]

be the set of counter examples to the makeability claim. Assume for the sake of contradiction that \( C \) is not empty. Then by the Well Ordering Principle, \( C \) must have some minimum element \( m \in C \).

First, observe that 12 is makeable using two 6¢ stamps. The next multiple of 3 is 15, which is makeable using a single stamp. So \( m \) is not 12 or 15, and since it is a multiple of 3, it must be as large as the next multiple of 3 after 15, namely, \( m \geq 18 \). This implies that \( m - 6 \) is multiple of 3 that is \( \geq 12 \) and less than \( m \). Since \( m \) is a minimum counterexample, \( m - 6 \) must be makeable. But then by adding a 6¢ stamp to the stamps that make up \( m - 6 \), we get a set of stamps that make \( m \)¢. This contradicts the fact that \( m \) is not makeable.

This contradiction implies that \( C \) must be empty. That is, all \( n \geq 12 \) that are multiples of 3 are makeable.

\[ \square \]

STAFF NOTE: A possible extension problem for further WOP practice is “Now characterize the makeable amounts of postage.” Here another WOP proof like the one for the WOP practice problem about 10 and 15 cent stamps verifies that all makeable numbers are multiples of three, so the makeable numbers are exactly the multiples of three \( \geq 12 \) along with the number 6.

STAFF NOTE: For the following problem, encourage your team to use the WOP proof template and get a complete solution written on their whiteboard. If someone brings up induction, confirm that induction would be a standard approach, but we’re focussing on WOP today and don’t want to be distracted by other approaches.

Problem 2.
Use the Well Ordering Principle \(^1\) to prove that

\[ \sum_{k=0}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

(1)

for all nonnegative integers, \( n \).

\(^1\)Proofs by other methods such as induction or by appeal to known formulas for similar sums will not receive credit.

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Solution. The proof is by contradiction.

Suppose to the contrary that equation (1) failed for some $n \geq 0$. Then by the WOP, there is a smallest nonnegative integer, $m$, such that (1) does not hold when $n = m$.

But (1) clearly holds when $n = 0$, which means that $m \geq 1$. So $m - 1$ is nonnegative, and since it is smaller than $m$, equation (1) must be true for $n = m - 1$. That is,

$$\sum_{k=0}^{m-1} k^2 = \frac{(m - 1)((m - 1) + 1)(2(m - 1) + 1)}{6}.$$  \hspace{1cm} (2)

Now add $m^2$ to both sides of equation (2). Then the left hand side equals

$$\sum_{k=0}^{m} k^2$$

and the right hand side equals

$$\frac{(m - 1)((m - 1) + 1)(2(m - 1) + 1)}{6} + m^2$$

Now a little algebra (given below) shows that the right hand side equals

$$\frac{m(m + 1)(2m + 1)}{6}.$$  \hspace{1cm} (3)

That is,

$$\sum_{k=0}^{m} k^2 = \frac{m(m + 1)(2m + 1)}{6},$$

contradicting the fact that equation (1) does not hold for $m$.

It follows that there is no smallest nonnegative integer for which equation (1) fails. Hence (1) must hold for all nonnegative integers.

Here’s the algebra:

$$\frac{(m - 1)((m - 1) + 1)(2(m - 1) + 1)}{6} + m^2 = \frac{(m - 1)m(2m - 1)}{6} + m^2$$

$$= \frac{(m^2 - m)(2m - 1)}{6} + m^2$$

$$= \frac{(2m^3 - 3m^2 + m)}{6} + \frac{6m^2}{6}$$

$$= \frac{(2m^3 + 3m^2 + m)}{6}$$

$$= \frac{m(m + 1)(2m + 1)}{6}$$

Problem 3.

Euler’s Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$
Integer values for $a, b, c, d$ that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two centuries to demonstrate his mistake.

Now let’s consider Lehman’s equation, similar to Euler’s but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4$$

(3)

Prove that Lehman’s equation (3) really does not have any positive integer solutions.

**Hint:** Consider the minimum value of $a$ among all possible solutions to (3).

**STAFF NOTE:** Students are sometimes sloppy about using WOP here and talk about “the minimum solution to equation (3).” But a solution is a vector $(a, b, c, d) \in \mathbb{Z}_+^4$, and the WOP does not apply directly. (There are various ways to salvage the idea of a minimum solution, for example, by considering the minimum value of the sum of the coordinates, but it’s best not to go there.)

**Solution.** Suppose that there exists a solution. Then there must be a solution in which $a$ has the smallest possible value. We will show that, in this solution, $a$, $b$, $c$, and $d$ must all be even. However, we can then obtain another solution over the positive integers with a smaller $a$ by dividing $a$, $b$, $c$, and $d$ in half. This is a contradiction, and so no solution exists.

All that remains is to show that $a$, $b$, $c$, and $d$ must all be even. The left side of Lehman’s equation is even, so $d^4$ is even, so $d$ must be even. Substituting $d = 2d'$ into Lehman’s equation gives:

$$8a^4 + 4b^4 + 2c^4 = 16d'^4$$

(4)

Now $2c^4$ must be a multiple of 4, since every other term is a multiple of 4. This implies that $c^4$ is even and so $c$ is also even. Substituting $c = 2c'$ into the previous equation gives:

$$8a^4 + 4b^4 + 32c'^4 = 16d'^4$$

(5)

Arguing in the same way, $4b^4$ must be a multiple of 8, since every other term is. Therefore, $b^4$ is even and so $b$ is even. Substituting $b = 2b'$ gives:

$$8a^4 + 64b'^4 + 32c'^4 = 16d'^4$$

(6)

Finally, $8a^4$ must be a multiple of 16, $a^4$ must be even, and so $a$ must also be even. Therefore, $a$, $b$, $c$, and $d$ must all be even, as claimed.

**Problem 4.**

You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

**Property** $m$: For any nonnegative integer less than $2^{m+1}$, there is a selection of envelopes whose contents add up to exactly that number of dollars.

Use the Well Ordering Principle (WOP) to prove that Property $m$ holds for all nonnegative integers $m$.

**Hint:** Consider two cases: first, when the target number of dollars is less than $2^m$ and second, when the target is at least $2^m$.

**STAFF NOTE:** Pedagogical advice: This is likely to be a challenging problem for beginners who have only seen WOP (or induction) used to prove algebraic formulas. It’s useful practice to have the team get a complete solution written on their whiteboard using the WOP template. (If someone brings up induction, confirm that induction would be a standard approach, but we’re focussing on WOP here and don’t want to be distracted by other approaches.)

You might get your team going by having them write out the WOP proof template to be filled in, and then get them focussed on case 1.
Students may be worried about how to invent this kind of split into cases. Tell them we don’t expect them to be able to do that yet, but now just want focus on using WOP to verify the claim. Afterward, you might try getting some team member to explain how binary numbers work, and have the team consider how that would be relevant:

**Further explanation:** An easy way to see why this holds is to think about \( m \) digit binary numbers. The binary representation of the number of dollars contained in a selection of envelopes has a 1 for its \( i \)th digit iff the \( i \)th envelope is in the selection (starting with an envelope numbered 0). This gives a recipe for finding a selection of envelopes whose contents add up to any given \( m \) digit binary number.

This is worth pointing out at some point, maybe before, or maybe after, students work out the WOP proof.

The solution below can be understood as first finding a selection of envelopes that fill in the low-order bits and then using (or not using) the last envelope to fill in the high order bit.

Thinking about binary numbers also suggests an alternative approach to the WOP proof based on first finding a selection for the high-order bits and then filling in the low order bit. Namely, let \( k := \lfloor n/2 \rfloor \) be the quotient of \( n \) divided by 2. This will be smaller than \( 2^{m_0} \), so Property \( m_0 - 1 \) implies that some selection of the first \( m_0 - 1 \) envelopes adds up to \( k \). Replacing each envelope in this selection by the next envelope (that contains twice as many dollars) gives a selection adding up to \( 2k \) that does not use the first envelope. If \( n = 2k \), we’re done, and if \( n = 2k + 1 \), then add the first envelope to the selection to get \( n \).

**Solution.** Let \( C \) be the set of positive integers \( m \) such that Property \( m \) is not true, and assume for the sake of contradiction that \( C \) is non-empty. Then by the Well Ordering Principle (WOP), \( C \) has a smallest element \( m_0 \).

The first thing to notice is that Property 0 holds, because the nonnegative integers less than \( 2^{0+1} \) are just one and zero, which we can always get by selecting or not selecting the single envelope containing one dollar. So \( m_0 \) must be greater than 0, and \( m_0 - 1 \) must be a nonnegative integer. The definition of \( m_0 \) now implies that Property \( m_0 - 1 \) must hold.

We will now prove that Property \( m_0 \) holds, contradicting the definition of \( m_0 \). To do this, we show how to find a selection of envelopes whose contents add up to any nonnegative integer \( n \) less than \( 2^{m_0+1} \).

There are two cases:

1. \( n < 2^{m_0} \). Now since \( 2^{m_0} = 2^{(m_0-1)+1} \) and Property \( m_0 - 1 \) holds, we can find a subset of the first \( m_0 - 1 \) envelopes that add up to \( n \). We don’t need to use the last envelope containing \( \$2^{m_0} \).

2. \( 2^{m_0} \leq n \). Since \( n < 2^{m_0+1} \), we know \( n - 2^{m_0} < 2^{m_0} \). Therefore by Case 1, there is a selection of the first \( m_0 - 1 \) envelopes that adds up to \( n - 2^{m_0} \) dollars. Now add the last envelope, which contains \( 2^{m_0} \) dollars, to this selection to obtain a selection that adds to \( n \) dollars.

Therefore, for any \( n < 2^{m_0+1} \) we can find a selection of envelopes whose contents add up to \( n \) dollars. This means that Property \( m_0 \) holds, contradicting the choice of \( m_0 \).

This contradiction implies that \( C \) must be empty, proving that Property \( m \) holds for all positive integers \( m \).

**Problem 5.**
Use the Well Ordering Principle to prove that any integer greater than or equal to 30 can be represented as the sum of nonnegative integer multiples of 6, 10, and 15.

**Hint:** Use the template for WOP proofs to ensure partial credit. Verify that integers in the interval \([30..35]\) are sums of nonnegative integer multiples of 6, 10, and 15.

**Solution.**
Claim 5.1. For all \( n \geq 30 \), it is possible to represent \( n \) as a sum of nonnegative integer multiples of 6, 10 and 15.

Proof. The proof is by the Well Ordering Principle. Let \( P(n) \) be the predicate that \( n \) is a sum of nonnegative integer multiples of 6, 10 and 15.

Let \( C := \{ n \geq 30 \mid \text{NOT}(P(n)) \} \) be the set of counter examples. Assume for the sake of contradiction that \( C \) is not empty. Then by the Well Ordering Principle, \( C \) must have some minimum element \( m \in C \).

First, observe that \( P(n) \) is true for the values of \( n \in [30..35] \).

- \( n = 30: 3 \cdot 10 \).
- \( n = 31: 6 + 15 + 10 \).
- \( n = 32: 2 \cdot 6 + 2 \cdot 10 \).
- \( n = 33: 3 \cdot 6 + 15 \).
- \( n = 34: 4 \cdot 6 + 10 \).
- \( n = 35: 15 + 2 \cdot 10 \).

We thus have \( m \geq 36 \). Since \( m \) is the smallest counterexample, \( m - 6 \) is not a counterexample. Since \( m - 6 \geq 30 \), it follows that \( P(m - 6) \) holds, that is, \( m - 6 \) as the sum of nonnegative integer multiples of 6, 10, and 15. Thus we can represent \( m \) by adding 1 to the coefficient of 6 in our representation of \( m - 6 \). This shows that \( m \) is not actually a counterexample, contradicting the assumption that \( C \) is nonempty. \( \blacksquare \)