Staff Solutions to In-Class Problems Week 2, Fri.

Problem 1.
For each of the logical formulas, indicate whether or not it is true when the domain of discourse is \( \mathbb{N} \) (the nonnegative integers \( 0, 1, 2, \ldots \)), \( \mathbb{Z} \) (the integers), \( \mathbb{Q} \) (the rationals), \( \mathbb{R} \) (the real numbers), and \( \mathbb{C} \) (the complex numbers). Add a brief explanation to the few cases that merit one.

\[
\begin{align*}
\exists x. x^2 &= 2 \\
\forall x. \exists y. x^2 &= y \\
\forall y. \exists x. x^2 &= y \\
\forall x \neq 0. \exists y. xy &= 1 \\
\exists x. \exists y. x + 2y &= 2 \text{ AND } 2x + 4y &= 5
\end{align*}
\]

**STAFF NOTE:** The few brief explanations for entries below are sufficient.

Intervene if teams start to go overboard with adding explanations (unlikely). After the problem has been team-approved (team check on their board), you can challenge a team member to provide an omitted explanation. If they had sufficient explanations (common), I like to challenge a team member with a “meta”-question, “Which was the hardest entry to fill in, and why?”

Solution.

<table>
<thead>
<tr>
<th>Statement</th>
<th>( \mathbb{N} )</th>
<th>( \mathbb{Z} )</th>
<th>( \mathbb{Q} )</th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists x. x^2 = 2 )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>( \forall x. \exists y. x^2 = y )</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>(( y = x^2 ))</td>
<td>T</td>
</tr>
<tr>
<td>( \forall y. \exists x. x^2 = y )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>(take ( y &lt; 0 ))</td>
</tr>
<tr>
<td>( \forall x \neq 0. \exists y. xy = 1 )</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>(( y = 1/x ))</td>
<td>T</td>
</tr>
<tr>
<td>( \exists x. \exists y. x + 2y = 2 \text{ AND } 2x + 4y = 5 )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Problem 2.
The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: \( \lambda, 0, 0, 0, 00, 01, 10, 11, 000, 001, \ldots \) (Here \( \lambda \) denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including =), variables, and the binary symbols 0, 1 denoting 0, 1.

A string like 01x0y of binary symbols and variables denotes the **concatenation** of the symbols and the binary strings represented by the variables. For example, if the value of \( x \) is 011 and the value of \( y \) is 1111, then the value of 01x0y is the binary string 0101101111.

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).
(a) $x$ consists of three copies of some string.  

**Solution.** $\exists y (x = yyy)$

(b) $x$ is an even-length string of 0’s.

**Solution.** $\text{NO-1S}(x) \land \exists y (x = yy)$

Some students mentioned $\lambda$ in their formulas. Technically, this is not allowed, so they need to justify it by giving a formula that means “$x = \lambda$.” This is easy, for example: $x = xx$.

A serious mistake was to try writing a recursive definition of a predicate calculus formula, as in

$$P(x) ::= x = \lambda \lor \exists y. x = 00y \land P(y).$$

(1)

Such recursive formulas are, by definition, not part of predicate calculus—with good reason. Definition (1) resembles a simple recursive definition of a procedure to test if $x$ is an even length string of 0’s, and its meaning might be explained in procedural terms. But it’s hard to figure out in general what recursively defined formulas mean. For example, what predicate is defined by the following “recursive definition”:

$$Q(n) ::= \text{NOT } Q(n)?$$

(c) $x$ does not contain both a 0 and a 1.

**Solution.** $\text{NOT}[\text{SUBSTRING}(0, x) \land \text{SUBSTRING}(1, x)]$

(d) $x$ is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

**Solution.** $(x = 10) \lor (\exists y (x = 1y1 \land \text{NO-1S}(y)))$

(e) An elegant, slightly trickier way to define $\text{NO-1S}(x)$ is:

$$\text{PREFIX}(x, \lambda x).$$

(*)

Explain why (*) is true only when $x$ is a string of 0’s.

**Solution.** Prefixing $x$ with 0 rightshifts all the bits. So the $n$th symbol of $x$ shifts into the $(n + 1)$st symbol of $0x$. Now for $x$ to be a prefix of $0x$, the $n$th symbol of $0x$ must match the $(n + 1)$st symbol of $x$. So if $x$ satisfies (*), the $n$th and $(n + 1)$st symbols of $x$ must match. This holds for all $n \geq 0$ up to the length of $x$, that is, all the symbols of $x$ must be the same. In addition, if $x \neq \lambda$, it must start with 0. Therefore, if $x$ satisfies (*), all its symbols must be 0’s.

Note that it’s easy to see, conversely, that if $x = \lambda$ or $x$ is all 0’s, then of course it satisfies (*).
**STAFF NOTE:** Explain why can’t we define “x is an even-length string of 0’s,” by

\[ \text{PREFIX}(x, 00x). \] (* *)

Problem 3.
Translate the following sentence into a predicate formula:

There is a student who has e-mailed at most two other people in the class, besides possibly himself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are

- equality, and
- \( E(x, y) \), meaning that “x has sent e-mail to y.”

**Solution.** A good way to begin tackling this problem is by trying to translate parts of the sentence. First of all, our formula must be of the form

\[ \exists x. P(x) \]

where \( P(x) \) should be a formula that says that “student x has e-mailed at most two other people in the class, besides possibly himself”.

One way to write \( P(x) \) is to write that “whenever we meet a student that has been e-mailed by x, this student is either x himself or y or z, where y and z are particular students in the class.”

To write the part “whenever we meet a student that has been e-mailed by x, this student is either x himself or y or z” we write

\[ \forall s. (E(x, s) \text{ IMPLIES } (s = x \text{ OR } s = y \text{ OR } s = z)) . \]

The part “where y and z are two students in the class” simply means that there exist two such students; so by adding the appropriate existential quantifiers, we get

\[ P(x) \coloneqq \exists y. \exists z. \forall s. (E(x, s) \text{ IMPLIES } (s = x \text{ OR } s = y \text{ OR } s = z)) . \]

At this point you may be thinking that \( P(x) \) says that “x has e-mailed exactly two students besides possibly himself.” However we did not require that y and z be distinct, or that they be different from x. So our formula describes all possibilities:

- \( x \) and exactly 2 other students: \( x \neq y, x \neq z, y \neq z. \)
- \( x \) and exactly 1 other student: \( x \neq y, x \neq z, y = z. \)
- \( x \) and no other student: \( x = y = z. \)

Overall the full formula is:

\[ \exists x. \exists y. \exists z. \forall s. (E(x, s) \text{ IMPLIES } (s = x \text{ OR } s = y \text{ OR } s = z)) . \] (2)

An alternate approach to defining the desired formula is arguably more straightforward than (2), but also more long-winded. Saying that ”student x has emailed at most two people besides himself” is the same as saying that ”x has not emailed three different, other people.”
Let’s give these other people names $a$, $b$ and $c$. Saying that $a$, $b$ and $c$ are “three different, other people” just means that $a$, $b$, $c$ and $x$ are all different. Saying that “$x$ has emailed each of these people” translates directly into a formula $R(x, a, b, c)$ of the required kind:

\[
R(x, a, b, c) \iff E(x, a) \text{ AND } E(x, b) \text{ AND } E(x, c) \text{ AND } (a \neq b) \text{ AND } (a \neq c) \text{ AND } (b \neq c) \text{ AND } (x \neq a) \text{ AND } (x \neq b) \text{ AND } (x \neq c).
\]

So the final formula we have been looking for is

\[
\exists x. \ \text{NOT}(\exists a, b, c. \ R(x, a, b, c)).
\]

The formulas (2) and (3) are different ways of expressing the same fact about student email recipients—they are equivalent formulas. The formula (2) is arguably a little easier to understand than (3), but it’s longer. Moreover, this difference in length becomes more dramatic if we consider generalizing from two students to $n$ students:

There is a student who has e-mailed at most $n$ other people in the class, besides possibly himself.

Now the approach of (3) leads to a formula with $n + 1$ variables all of which are supposed to designate different students. We say they are all different by including about $n^2/2$ “not-equal” formulas between variables. On the other hand, the approach of (2) leads to a formula of overall size proportional to $n$.

Problem 4.
Provide a counter model for the implication that is not valid. Informally explain why the other one is valid.

1. $\forall x. \exists y. P(x, y) \iff \exists y. \forall x. P(x, y)$

2. $\exists y. \forall x. P(x, y) \iff \forall x. \exists y. P(x, y)$

Solution. The first implication, $\forall x. \exists y. P(x, y) \implies \exists y. \forall x. P(x, y)$, is invalid.

One counter model is the predicate $P(x, y) \iff y < x$ where the domain of discourse is the real numbers, $\mathbb{R}$. For every real number $x$, there exists a real number $y$ which is strictly less than $x$, so the antecedent of the implication is true. But there is no minimum real number, so the consequent is false.

The second implication is valid. Let’s say that “$x$ is good for $y$” when $P(x, y)$ is true. The hypothesis says that there is some element, call it $g$, that is good for everything. The conclusion is that every element has something that is good for it, which of course is true since $g$ will be good for it.

**STAFF NOTE:** It’s not clear students will be able to articulate the validity explanation. If they get stuck, offer them the “$x$ is good for $y$” phrase as helpful. If it doesn’t help, then explain the answer.

Supplemental Problem

**STAFF NOTE:** This problem is a fun puzzle that provides more practice with predicate formulas. Tell students it’s here for backup—we don’t really expect there will be time for it.

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1There is no need to study supplemental problems when preparing for quizzes or exams.
Problem 5.
A certain cabal within the Math for Computer Science course staff is plotting to make the final exam ridiculously hard. (“Problem 1. Prove the Poincare Conjecture starting from the axioms of ZFC. Express your answer in khipu—the knot language of the Incas.”) The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of seven people:

{Adam, Tom, Albert, Annie, Ben, Elizabeth, Siggi}

The cabal is a subset of these seven. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate cabal indicates who is in the cabal; that is, cabal(x) is true if and only if x is a member. Translate each statement below into English and deduce who is in the cabal.

(a) $\exists x, y, z. (x \neq y \land x \neq z \land y \neq z \land \text{cabal}(x) \land \text{cabal}(y) \land \text{cabal}(z))$

Solution. A direct English paraphrase would be “There exist people we’ll call x, y, and z, who are all different, such that x, y and z are each in the cabal.” A better version would use the fact that there’s no need in this case to give names to the people. Namely, a better paraphrase is “There are at least 3 different people in the cabal.” Perhaps a simpler way to say this is: “The cabal is of size at least 3.”

(b) $\neg(\text{cabal}(\text{Siggi}) \land \text{cabal}(\text{Annie}))$

Solution. Siggi and Annie are not both in the cabal. Equivalently: at least one of Siggi and Annie is not in the cabal.

(c) $\text{cabal}(\text{Elizabeth}) \Rightarrow \forall x. \text{cabal}(x)$

Solution. If Elizabeth is in the cabal, then everyone is.

(d) $\text{cabal}(\text{Annie}) \Rightarrow \text{cabal}(\text{Siggi})$

Solution. If Annie is in the cabal, then Siggi is also.

(e) $(\text{cabal}(\text{Ben}) \lor \text{cabal}(\text{Albert})) \Rightarrow \neg \text{cabal}(\text{Tom})$

Solution. If either of Ben or Albert is in the cabal, then Tom is not. Equivalently, if Tom is in the cabal, then neither Albert nor Ben is.

(f) $(\text{cabal}(\text{Ben}) \lor \text{cabal}(\text{Siggi})) \Rightarrow \neg \text{cabal}(\text{Adam})$

Solution. If either of Ben or Siggi is in the cabal, then Adam is not. Equivalently, if Adam is in the cabal, the neither Ben nor Siggi is.

(g) Now use these facts to explain exactly who is on the cabal and why.

STAFF NOTE: If a team is stuck, tell them that the cabal consists of exactly Siggi, Ben, and Albert and have them check that this set satisfies all the conditions. (See the end of the solution.) Then start them back on proving that this is the unique set that works.
Solution. So much for the translations. We now argue that the only cabal satisfying all six propositions above is one whose members are exactly Siggi, Ben, and Albert.

We first observe that by (b), there must be someone—either Siggi or Annie—who is not in the cabal. But if Elizabeth were in the cabal, then by (c), everyone would be. So we conclude by contradiction that:

Elizabeth is not in the cabal. \hspace{1cm} (4)

Next observe that if Annie was in the cabal, then by (d), Siggi would be too, contradicting (b). So by again contradiction, we conclude:

Annie is not in the cabal. \hspace{1cm} (5)

Now suppose Tom is in the cabal. Then by (e), Ben and Albert are not, and we already know Elizabeth and Annie are not, so only three remain who could be in the cabal, namely, Tom, Siggi, and Adam. But by (a) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

**Lemma 5.1** (TNE). *If Tom is in the cabal, then Siggi and Adam are in the cabal.*

But by (f), if Siggi is in the cabal, then Adam is not. That is,

**Lemma 5.2** (NnE). *Siggi and Adam cannot both be in the cabal.*

Now from Lemma (NnE) we conclude that the conclusion of Lemma (TNE) is false. So by contrapositive, the hypothesis of Lemma (TNE) must also be false, namely,

Tom is not in the cabal. \hspace{1cm} (6)

Finally, suppose Adam is in the cabal. Then by (f), Ben and Siggi are not, and we already know Elizabeth, Annie and Tom are not. So the cabal must consist of at most two people (Albert and Adam). This contradicts (a), and we conclude by contradiction that

Adam is not in the cabal. \hspace{1cm} (7)

So the only remaining people who could be in the cabal are Albert, Ben, and Siggi. Since the cabal must have at least three members, we conclude that

**Lemma.** *The only possible cabal consists of Albert, Ben, and Siggi.*

But we’re not done yet: we haven’t shown that a cabal consisting of Albert, Ben, and Siggi actually does satisfy all six conditions. So let \( A = \{ \text{Albert, Ben, Siggi} \} \), and let’s quickly check that \( A \) satisfies (a)--(f):

- \( |A| = 3 \), so \( A \) satisfies (a).
- Annie is not in \( A \), so \( A \) satisfies (b) and (d).
- Elizabeth is not in \( A \), so the hypothesis of (c) is false, which means that \( A \) satisfies (e).
- Finally, Tom and Adam are not in \( A \), so the conclusions of both (e) and (f) are true, and so \( A \) satisfies (e) and (f).

So now we have proved

**Proposition.** *\{Albert, Ben, Siggi\} is the unique cabal satisfying conditions (a)--(f).*