Staff Solutions to In-Class Problems Week 13, Wed.

STAFF NOTE: Expectation, Ch. 19.4-19.5

Problem 1.
Here’s a dice game with maximum payoff $k$: make three independent rolls of a fair die, and if you roll a six
- no times, then you lose 1 dollar;
- exactly once, then you win 1 dollar;
- exactly twice, then you win 2 dollars;
- all three times, then you win $k$ dollars.

For what value of $k$ is this game fair?\(^1\)

Solution. Let the random variable $P$ be your payoff. We can compute $\text{Ex}[P]$ as follows:

\[
\text{Ex}[P] = -1 \cdot \text{Pr}[0 \text{ sixes}] + 1 \cdot \text{Pr}[1 \text{ six}] + 2 \cdot \text{Pr}[2 \text{ sixes}] + k \cdot \text{Pr}[3 \text{ sixes}]
\]
\[
= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3 \cdot \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 + 2 \cdot 3 \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) + k \cdot \left(\frac{1}{6}\right)^3
\]
\[
= \frac{-125 + 75 + 30 + k}{216}
\]

The game is fair when $\text{Ex}[P] = 0$. This happens when $k = 20$.

Problem 2.
A classroom has sixteen desks in a $4 \times 4$ arrangement as shown below.

\[\begin{array}{cccc}
\text{desk} & \text{desk} & \text{desk} & \text{desk} \\
\text{desk} & \text{desk} & \text{desk} & \text{desk} \\
\text{desk} & \text{desk} & \text{desk} & \text{desk} \\
\text{desk} & \text{desk} & \text{desk} & \text{desk}
\end{array}\]

\(^1\)This game is actually offered in casinos with $k = 3$, where it is called Carnival Dice.
If there is a girl in front, behind, to the left, or to the right of a boy, then the two flirt. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied mutually independently by boys and girls with equal probability. What is the expected number of flirting couples?  

*Hint*: Linearity.

**Solution.** A natural first approach to this problem is to calculate the expected number of flirtations that each desk is involved in, add the expectations for each desk, and then divide by two (since each flirtation involves two desks). This approach works fine, but it requires finding the expectations for three different kinds of desks: corner, side, and middle.

A more elegant approach is to note that the expected number of flirtations between adjacent desks is $1/2$, and the number of pairs of adjacent desks is 24—there are 12 pairs adjacent horizontally (3 in each of 4 rows) and likewise 12 pairs adjacent vertically. So by linearity of expectation, the expected number of flirtations is $(1/2) \cdot 24 = 12$.

To be more explicit about this application of linearity, let’s arbitrarily number the pairs of adjacent desks from 1 to 24 and let $F_i$ be an indicator random variable for the event that occupants of the desks in the $i$-th pair are flirting. The occupants of adjacent desks are flirting iff they are of opposite sexes, which happen with probability 1/2, so $\Pr[F_i = 1] = 1/2$. The expectation of an indicator variable is the same as the probability that it equals 1, so

$$\text{Ex}[F_i] = \frac{1}{2}. \quad (1)$$

Now, if $F$ is the number of flirting couples, then $F = \sum_{i=1}^{24} F_i$, so the expectation we want is

$$\text{Ex}[F] = \text{Ex} \left[ \sum_{i=1}^{24} F_i \right]$$

$$= \sum_{i=1}^{24} \text{Ex}[F_i] \quad \text{(linearity of Ex[:])}$$

$$= \sum_{i=1}^{24} \frac{1}{2} \quad \text{(equation (1))}$$

$$= 24 \cdot \frac{1}{2} = 12.$$
Problem 3. (a) Suppose we flip a fair coin and let $N_{TT}$ be the number of flips until the first time two consecutive Tails appear. What is $\text{Ex}[N_{TT}]$?

*Hint:* Let $D$ be the tree diagram for this process. Explain why $D$ can be described by the tree in Figure 1. Use the Law of Total Expectation: Let $R$ be a random variable and $A_1, A_2, \ldots$, be a partition of the sample space. Then

$$
\text{Ex}[R] = \sum_i \text{Ex}[R \mid A_i] \text{Pr}[A_i].
$$

**STAFF NOTE:** Ask what’s wrong with the following argument: *the probability of flipping two heads in a row in 1/4, so by the mean time to failure rule, $\text{Ex}[N_{TT}] = 1/(1/4) = 4$, contradicting the value 6 derived above.*

The answer is that MTF applies only if the probability of failure at each step is the same *independent* of the previous flips, but that is obviously not true here, since the probability is twice as large if the previous flip was $T$ than if it was $H$.

**Solution.**

$$
\text{Ex}[N_{TT}] = 6.
$$

Let $H$ be the event that a Head appears on the first flip, $TH$ the event that the first flips are Tail then Head, and likewise $TT$. From $D$ and the Law of Total Expectation:

$$
\text{Ex}[N_{TT}] = \text{Ex}[N_{TT} \mid H] \cdot \text{Pr}[H] + \text{Ex}[N_{TT} \mid TH] \cdot \text{Pr}[TH] + \text{Ex}[N_{TT} \mid TT] \cdot \text{Pr}[TT]
$$

$$
= (1 + \text{Ex}[N_{TT}]) \cdot \frac{1}{2} + (2 + \text{Ex}[N_{TT}]) \cdot \frac{1}{4} + 2 \cdot \frac{1}{4}
$$

$$
= \frac{1}{2} + \frac{\text{Ex}[N_{TT}]}{2} + \frac{1}{2} + \frac{\text{Ex}[N_{TT}]}{4} + \frac{1}{2}
$$

$$
= \frac{3}{2} + 3 \frac{\text{Ex}[N_{TT}]}{4}
$$

So

$$
\text{Ex}[N_{TT}] = \frac{3}{2} \cdot 4 = 6.
$$

(b) Let $N_{TH}$ be the number of flips until a Tail immediately followed by a Head comes up. What is $\text{Ex}[N_{TH}]$?
Solution.

\[ \text{Ex}[N_{TH}] = 4. \]

This time the tree diagram is \( C := H \cdot C + T \cdot B \), where the subtree \( B := H + T \cdot B \).

So

\[ \text{Ex}[N_{TH}] = (1 + \text{Ex}[N_{TH}]) \cdot \frac{1}{2} + (1 + \text{Ex}[N_B]) \cdot \frac{1}{2} \]

where \( N_B \) is the expected number of flips in the \( B \) subtree. But

\[ \text{Ex}[N_B] = 1 \cdot \frac{1}{2} + (1 + \text{Ex}[N_B]) \cdot \frac{1}{2}. \]

That is, \( \text{Ex}[N_B] = 2 \). Hence,

\[ \text{Ex}[N_{TH}] = \frac{1}{2} + \frac{\text{Ex}[N_{TH}]}{2} + \frac{1}{2} + \frac{2}{2} \]

which implies \( \text{Ex}[N_{TH}] = 4. \)

\( \blacksquare \)

(c) Suppose we now play a game: flip a fair coin until either \( TT \) or \( TH \) occurs. You win if \( TT \) comes up first, and lose if \( TH \) comes up first. Since \( TT \) takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you’re willing to play if you pay him $5 when he wins, and he pays you with a mere 20% premium—that is $6—when you win.

If you do this, you’re sneakily taking advantage of your opponent’s untrained intuition, since you’ve gotten him to agree to unfair odds. What is your expected profit per game?

**STAFF NOTE:** After the problem is solved, start a discussion of the apparent paradox: \( TT \) and \( TH \) are equally likely to show up first, but \( TT \) takes longer to show up on average.

**Solution.** It’s easy to see that both \( TT \) and \( TH \) are equally likely to show up first: every game play consists of a sequence of \( H \)’s followed by a \( T \), after which the game ends with a \( T \) or an \( H \), with equal probability. So your expected profit is

\[ \frac{1}{2} \cdot 6 + \frac{1}{2} \cdot (-5) \]

dollars, that is 50 cents per game.

It may seem paradoxical that \( TT \) and \( TH \) are equally likely to show up first, but \( TT \) takes longer to show up on average. The explanation is that \( TT \) takes longer to show up after \( TH \) has appeared than \( TH \) takes to show up after \( TT \) has appeared. That’s because when \( TT \) appears, we’re already one step along the way to having \( TH \) appear afterward, yet when \( TH \) shows up first, we have to start waiting for \( TT \) to appear without a similar head start.

\( \blacksquare \)

**Problem 4.**

Let \( T \) be a positive integer valued random variable such that

\[ \text{PDF}_T(n) = \frac{1}{an^2}, \]

where

\[ a := \sum_{n \in \mathbb{Z}^+} \frac{1}{n^2}. \]

(a) Prove that \( \text{Ex}[T] \) is infinite.
Solution.

\[ \text{Ex}[T] := \sum_{n \in \mathbb{Z}^+} n \text{PDF}_T(n) \]
\[ = \sum_{n \in \mathbb{Z}^+} n \frac{1}{an^2} \]
\[ = \sum_{n \in \mathbb{Z}^+} \frac{1}{an} \]
\[ = \frac{1}{a} \lim_{n \to \infty} H_n = \infty. \]

(b) Prove that \( \text{Ex}[\sqrt{T}] \) is finite.

Solution.

\[ \text{Ex}[\sqrt{T}] = \sum_{n \in \mathbb{Z}^+} \sqrt{n} \cdot \frac{1}{an^2} \]
\[ = \sum_{n \in \mathbb{Z}^+} \frac{1}{an^{3/2}} \]
\[ < \frac{1}{a} \left( 1 + \int_1^{\infty} \frac{1}{n^{3/2}} \right) \]
\[ = \frac{1}{a} + \frac{1}{a} \left( -2 \sqrt{n} \right) \]
\[ = \frac{1}{a} + \frac{1}{a} \left( 0 - \frac{-2}{\sqrt{n}} \right) \]
\[ = \frac{3}{a} < \infty \]

**STAFF NOTE:** Note that if we define \( R = \sqrt{T} \), then \( R \) has finite expectation, but the variance of \( R \) is infinite.
Supplementary Problems

Problem 5.
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1, 2, 3, and 4.
Each desk is occupied by a male with probability $p > 0$ or a female with probability $q := 1 - p > 0$. A male and a female flirt when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.
(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

Solution. If $p = q = 1/2$ then $\Pr[I_{12} = 1] = \Pr[I_{23} = 1] = 1/2$ and $\Pr[I_{12} = 1 & I_{23} = 1]$ can be calculated from the fact that only F-M-F and M-F-M are possible when both couples are flirting. In that case, we have $\Pr[I_{12} = 1 & I_{23} = 1] = 2/8 = 1/4 = \Pr[I_{12} = 1] \cdot \Pr[I_{12} = 1]$. 

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

Solution. We can again compare $\Pr[I_{12} = 1 & I_{23} = 1]$ and $\Pr[I_{12} = 1] \cdot \Pr[I_{23} = 1]$.
As in the previous part, $I_{12} = 1 & I_{23} = 1$ only happen when we have a pattern of F-M-F or M-F-M for students 1, 2, and 3 respectively. These occur with total probability $p^2q + pq^2$. On the other hand, $I_{12}$ happens with probability $2pq$ total, accounting for the two patterns possible, M-F and F-M. Hence, $I_{12}$ and $I_{23}$ are independent iff $p^2q + pq^2 = pq(p + q) = 4p^2q^2$. By manipulating the expression we get $p + q = 4pq$. Recall $p + q = 1$. Hence, we are dealing with $1 = 4p - 4p^2$. The equation can be factored into $(2p - 1)^2 = 0$, yielding $p = 1/2$.

(c) What is the expected number of flirting couples in terms of $p$ and $q$?

Solution. The expected number of couples is $8pq$ by linearity of expectation.

Problem 6.
Justify each line of the following proof that if $R_1$ and $R_2$ are independent, then

$$\text{Ex}[R_1 \cdot R_2] = \text{Ex}[R_1] \cdot \text{Ex}[R_2].$$
Proof.

\[
\begin{align*}
\text{Ex}[R_1 \cdot R_2] &= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \\
&= \sum_{r_1, r_2 \in \text{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \left( r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1].
\end{align*}
\]

\[\square\]

Solution. Note that the event \([R_1 \cdot R_2 = r]\) is the disjoint union of events \([R_1 = r_1 \text{ and } R_2 = r_2]\) such that \(r_i \in \text{range}(R_i)\) for \(i = 1, 2\) and \(r_1 r_2 = r\).

Proof.

\[
\begin{align*}
\text{Ex}[R_1 \cdot R_2] &::= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \quad \text{(by definition)} \\
&= \sum_{r_1, r_2 \in \text{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \quad \text{(remarked above)} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \quad \text{(ordering terms in the sum)} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \quad \text{(independence of } R_1, R_2) \\
&= \sum_{r_1 \in \text{range}(R_1)} \left( r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \quad \text{(factor out } r_1 \Pr[R_1 = r_1]) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \quad \text{(def of Ex}[R_2]) \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \quad \text{(factor out Ex}[R_2]) \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1]. \quad \text{(def of Ex}[R_1])
\end{align*}
\]

\[\square\]