Staff Solutions to In-Class Problems Week 12, Mon.

STAFF NOTE: Independence, 18.7-18.8

Problem 1.
Sally Smart just graduated from high school. She was accepted to three reputable colleges.

- With probability $\frac{4}{12}$, she attends Yale.
- With probability $\frac{5}{12}$, she attends MIT.
- With probability $\frac{3}{12}$, she attends Little Hoop Community College.

Sally is either happy or unhappy in college.

- If she attends Yale, she is happy with probability $\frac{4}{12}$.
- If she attends MIT, she is happy with probability $\frac{7}{12}$.
- If she attends Little Hoop, she is happy with probability $\frac{11}{12}$.

(a) A tree diagram to help Sally project her chance at happiness is shown below. On the diagram, fill in the edge probabilities, and at each leaf write the probability of the corresponding outcome.

Solution. See Figure 1.

(b) What is the probability that Sally is happy in college?
Solution. The probability that Sally is happy is equal to the sum of the probabilities of the outcomes marked with an “H”: \( \frac{16}{144} + \frac{35}{144} + \frac{33}{144} = \frac{84}{144} = \frac{7}{12} \).

(c) What is the probability that Sally attends Yale, given that she is happy in college?

Solution.

\[
\Pr[\text{attends Yale} \mid \text{happy}] = \frac{\Pr[\text{attends Yale} \cap \text{happy}]}{\Pr[\text{happy}]} = \frac{(4/12) \cdot (4/12)}{(7/12)} = \frac{4}{21}
\]

(d) Show that the event that Sally attends Yale is not independent of the event that she is happy.

Solution. These two events are independent only if

\[
\Pr[\text{attends Yale} \mid \text{happy}] = \Pr[\text{attends Yale}]
\]

or \( \Pr[\text{happy}] = 0 \). However, the left side is \( \frac{4}{21} \), the right side is \( \frac{4}{12} \), and the probability that Sally is happy is nonzero.

(e) Show that the event that Sally attends MIT is independent of the event that she is happy.

Solution. These two events are independent if

\[
\Pr[\text{attends MIT}] \cdot \Pr[\text{happy}] = \Pr[\text{attends MIT} \cap \text{happy}]
\]

The left side is equal to \( \frac{5}{12} \cdot \frac{7}{12} = \frac{35}{144} \). According to the tree diagram, the right side is equal to \( \frac{35}{144} \) as well.

Problem 2.

Define the following events as in Section 18.6:

- \( A \): a random applicant is admitted,
- \( F_{EE} \): the applicant is a woman for the EE department,
- \( F_{CS} \): the applicant is a woman for the CS department,
- \( M_{EE} \): the applicant is a man for the EE department,
- \( M_{CS} \): the applicant is a man for the CS department,

where all applicants are assumed to be either men or women, and no applicants is in both departments.

In these terms, the plaintiff in a discrimination suit against a university makes the argument that in both departments, the probability that a female is admitted is less than the probability for a male. That is,

\[
\Pr[A \mid F_{EE}] < \Pr[A \mid M_{EE}]
\]  \hspace{1cm}

and

\[
\Pr[A \mid F_{CS}] < \Pr[A \mid M_{CS}].
\]

(1)  \hspace{1cm} (2)
The university’s defence attorneys retort that overall, a female applicant is more likely to be admitted than a male, namely, that

\[
\Pr[A \mid F_{EE} \cup F_{CS}] > \Pr[A \mid M_{EE} \cup M_{CS}].
\]  

(3)

The judge then interrupts the trial and calls the plaintiff and defence attorneys to a conference in his office to resolve what he thinks are contradictory statements of facts about the admission data. The judge points out that:

\[
\begin{align*}
\Pr[A \mid F_{EE} \cup F_{CS}] &= \Pr[A \mid F_{EE}] + \Pr[A \mid F_{CS}] \\
&< \Pr[A \mid M_{EE}] + \Pr[A \mid M_{CS}] \\
&= \Pr[A \mid M_{EE} \cup M_{CS}] \\
\end{align*}
\]

(because \(F_{EE}\) and \(F_{CS}\) are disjoint)

(4)

so

\[
\Pr[A \mid F_{EE} \cup F_{CS}] < \Pr[A \mid M_{EE} \cup M_{CS}].
\]

which directly contradicts the university’s position (3)!

Of course the judge is mistaken; an example where the plaintiff and defence assertions are all true appears in the text. What is the mistake in the judge’s proof?

**Solution.** The two equalities asserted by the judge are based on:

**False Claim.** If \(B\) and \(C\) are disjoint events, then for any event \(A\),

\[
\Pr[A \mid B \cup C] = \Pr[A \mid B] + \Pr[A \mid C].
\]  

(4)

But (4) is false in general. For example, let \(B\) and \(C\) be any disjoint events that both have nonzero probability, and let \(A := B \cup C\). Then

\[
\Pr[A \mid B \cup C] = 1 \neq 1 + 1 = \Pr[A \mid B] + \Pr[A \mid C].
\]

In this particular case, the identity

\[
\Pr[A \mid F_{EE} \cup F_{CS}] = \Pr[A \mid F_{EE}] + \Pr[A \mid F_{CS}]
\]

would fail if more than half of all female applicants were admitted in each department, since then the right hand side of this equation would be greater than 1.

Problem 3.

**Graphs, Logic & Probability** Let \(G\) be a simple graph. Let \(E(x, y)\) mean that \(G\) has an edge between vertices \(x\) and \(y\). Since \(G\) is simple, we have \(E(x, y) \iff E(y, x)\) and also \(\neg (E(x, x))\).

Let \(W(x, y)\) mean that there is a length-two walk in \(G\) between \(x\) and \(y\).

(a) Explain why \(E(x, y)\) implies \(W(x, y)\).

**Solution.** Going back and forth on edge \((x—y)\) is a length-two walk from \(x\) to \(x\).

(b) Write a predicate-logic formula defining \(W(x, y)\) in terms of \(E(u, v)\).

**Solution.**

\[
W(x, y) ::= \exists z. E(x, z) \text{ AND } E(z, y).
\]
Now let $V$ be a fixed set of vertices and $G$ be a graph with $V(G) = V$ constructed randomly as follows: for every two distinct vertices $u, v \in V$, independently include edge $(u-v)$ in $E(G)$ with probability $p$.

So $Pr[E(u, v)] = p$, and $E(u, v)$ is independent of $E(r, s)$ when $(u-v) \neq (r-s)$.

**STAFF NOTE:** Formally, our sample space is the set of simple graphs $G$ with $V(G) = V$ and $Pr[G] = p^k (1-p)^m$, where $k = |E(G)|$ and $m = \binom{n}{2} - k$.

An event $E$ **depends on an edge** $(u-v)$ iff $E$ and $E(u, v)$ are not independent. If two events **depend on disjoint sets of edges**, then they are independent.

(c) Let $g$ be the number of edges of $G$. Express $Pr[G]$ in terms of $g, n, p$.

**Solution.**

$$p^g (1-p)^{\binom{n}{2}-g}.$$  

In parts (d)–(f) below, circle the event pairs below that are independent. In each case, briefly explain your answer.

(d) $E(x, y)$ versus $W(x, y)$ for $x \neq y$.

**Solution. True.** Since there are no self-loops, the edge $(x-y)$ cannot be part of any length-two walk from $x$ to $y$. So the existence of a length-two walk from $x$ to $y$ does not depend on whether this edge is present in $G$.

(e) $[E(x, w) \text{ AND } E(w, y)]$ versus $[E(x, z) \text{ AND } E(z, y)]$ for distinct vertices $w, x, y$ and $z$.

**Solution. True.** Since edges are chosen independently and since all the following edges are distinct, the presence of edges $(x-w), (w-y)$, is independent of the presence of edges $(x-z), (z-y)$.  

(f) $W(w, x)$ versus $W(y, z)$ for distinct vertices $w, x, y$ and $z$.

**Solution. False.** The more randomly chosen edges there are in $G$, the more likely length two walks become. Now $W(w, x)$ implies the existence of two edges—which might be incident to $y$ or $z$—and this makes $W(y, z)$ more likely given $W(w, x)$.

(g) Write a simple formula in terms of $n$ and $p$ for $Pr[\overline{W(x, y)}]$, for distinct vertices $x$ and $y$.

**Hint:** Use part (e).

**Solution.** Let $Z := V - \{x, y\}$ be the set of all the vertices other than $x$ and $y$.

$$Pr[\overline{W(x, y)}] = Pr[\overline{\text{AND}_{z \in Z} \overline{E(x, z) \text{ AND } E(z, y)}}],$$

$$= \prod_{z \in Z} Pr[\overline{E(x, z) \text{ AND } E(z, y)}], \quad \text{(indep. by part (e))}$$

$$= \prod_{z \in Z} (1 - Pr[E(x, z) \text{ AND } E(z, y)]),$$

$$= \prod_{z \in Z} (1 - Pr[E(x, z)] \cdot Pr[E(y, z)]), \quad \text{(edges are independent)}$$

$$= \prod_{z \in Z} (1 - p^2),$$

$$= (1 - p^2)^{n-2}.$$
(h) Express $x$ and $y$ being on a three-cycle as a simple predicate formula involving $E(x, y)$ and $W(x, y)$.

Solution. $x$ and $y$ lie on a three-cycle iff $E(x, y)$ AND $W(x, y)$.

(i) Let $r := \Pr[\text{NOT}(W(x, y))]$. Write a simple expression in terms of $p$ and $r$ for the probability that $x$ and $y$ lie on a three-cycle in $G$.

Hint: Parts (d) and (h).

Solution.

$$p(1 - r).$$

Since $E(x, y)$ and $W(x, y)$ are independent,

$$\Pr[E(x, y) \text{ AND } W(x, y)] = \Pr[E(x, y)] \cdot \Pr[W(x, y)]$$

$$= p(1 - r).$$

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**Supplemental Problem**

**Problem 4.**

Describe events $A$, $B$, and $C$ that:

- satisfy the “product rule,” namely,

  $$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C],$$

- no two out of the three events are independent.

Hint: Choose $A$, $B$, $C$ events over the uniform probability space on $[1..6]$.

Solution.

$$A := \{1, 2, 3\}, \quad B := \{3, 4, 5\}, \quad C := \{3, 4, 5, 6\}.$$ 

So

$$\Pr[A \cap B \cap C]$$

$$= \Pr[3] = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3}$$

$$= \Pr[A] \cdot \Pr[B] \cdot \Pr[C],$$

$$\Pr[A \cap B] = \Pr[3] = \frac{1}{6}$$

$$\neq \frac{1}{2} \cdot \frac{1}{2} = \Pr[A] \cdot \Pr[B],$$

$$\Pr[A \cap C] = \Pr[3] = \frac{1}{6}$$

$$\neq \frac{1}{2} \cdot \frac{2}{3} = \Pr[A] \cdot \Pr[C],$$

$$\Pr[B \cap C] = \Pr[5] = \frac{1}{2}$$

$$\neq \frac{1}{2} \cdot \frac{2}{3} = \Pr[B] \cdot \Pr[C].$$