Problem Set 9 Solutions

Due: Wednesday, November 12

Reading assignment: Sections 11.8–11.11, 12.1–12.4, 12.6

Problem 1. [10 points]

(a) [5 pts] Show that of any \( n + 1 \) distinct numbers chosen from the set \( \{1, 2, \ldots, 2n\} \), at least 2 must be relatively prime. (Hint: \( \gcd(k, k + 1) = 1 \).)

Solution. Treat the \( n + 1 \) numbers as the pigeons and the \( n \) disjoint subsets of the form \( \{2j - 1, 2j\} \) as the pigeonholes. Then there must exist a pair of consecutive integers among the \( n + 1 \) chosen, which must be relatively prime (as suggested).

(b) [5 pts] Show that any finite connected undirected graph with \( n \geq 2 \) vertices must have 2 vertices with the same degree.

Solution. In a finite connected graph with \( n \geq 2 \) vertices, the domain for the vertex degrees is the set \( \{1, 2, \ldots, n - 1\} \) since each vertex can be adjacent to at most all of the remaining \( n - 1 \) vertices and the existence of a degree 0 vertex would violate the requirement that the graph be connected. Therefore, treating the \( n \) vertices as pigeons and the \( n - 1 \) possible degrees as pigeonholes, there must exist a pair of vertices with the same degree.

Problem 2. [15 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences “18062”, “6042” or “35876” (his MIT extension).

How many 10-digit passwords can he pick that don’t contain these forbidden sequences if each digit 0, 1, \ldots, 9 can only be chosen once (i.e. without repeats)?

Solution. Our general approach follows the inclusion–exclusion principle. The number of passwords he can choose is the number of permutations of the 10 digits minus the number of passwords containing one or more of the forbidden sequences.

There are 6 positions in which 18062 could appear, and the remaining digits could be any permutation of the remaining 5 digits. Therefore, there are \( 6 \cdot 5! \) passwords containing
Problem 3. [15 points] Give a combinatorial proof of the following theorem:

\[ n^{2^{n-1}} = \sum_{k=1}^{n} k \binom{n}{k} \]

(Hint: Consider the set of all length-\(n\) sequences of 0’s, 1’s and a single \(*\).)

**Solution.** Let \( P = \{0, \ldots, n-1\} \times \{0, 1\}^{n-1} \), and let \( S \) be the set of sequences as specified in the problem. On the one hand, there is a bijection from \( P \) to \( S \) by mapping \((k, x)\) to the word obtained by inserting a \(*\) just after the \(k\)th bit in the length-\(n-1\) binary word, \(x\). So

\[ |S| = |P| = n^{2^{n-1}} \quad (1) \]

by the Product Rule.

On the other hand, every sequence in \( S \) contains between 1 and \(n\) nonzero entries since the \(*\), at least, is nonzero. The mapping from a sequence in \( S \) with exactly \(k\) nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the \(*\) among these entries is a bijection, and the number of such pairs is \(\binom{n}{k}k\) by the Generalized Product Rule. Thus, by the Sum Rule:

\[ |S| = \sum_{k=1}^{n} k \binom{n}{k} \]

Equating this expression and the expression (1) for \(|S|\) proves the theorem.

Problem 4. [20 points] Calculate the following. Make sure to explain your reasoning.

(a) [5 pts] How many \(n\)-digit PIN numbers are there where no 2 consecutive digits are the same?
Solution. There are 10 choices for the first digit, and 9 choices for each of the remaining \( n - 1 \) digits, since you can choose any digit that is not the same as the one right in front of it, so there are

\[ 10 \cdot 9^{n-1} \]

such PIN numbers.

(b) [5 pts] How many numbers in the range \([1..700]\) are divisible by 2, 5 or 7?

Solution. Let \( S \) be the set of all numbers in range \([1..700]\). Let \( S_2 \) be the numbers in this range that are divisible by 2, let \( S_5 \) be the numbers in this range that are divisible by 5, and let \( S_7 \) be the numbers in this range that are divisible by 7. By inclusion–exclusion, the number of elements in \( S \) that are divisible by 2, 5 or 7 is

\[
n = |S_2| + |S_5| + |S_7| - |S_2S_5| - |S_2S_7| - |S_5S_7| + |S_2S_5S_7| \\
= \frac{700}{2} + \frac{700}{5} + \frac{700}{7} - \frac{700}{2 \cdot 5} - \frac{700}{2 \cdot 7} - \frac{700}{5 \cdot 7} + \frac{700}{2 \cdot 5 \cdot 7} \\
= 350 + 140 + 100 - 70 - 50 - 20 + 10 \\
= 460.
\]

(c) [10 pts] How many 10 digit numbers are there in which there are exactly 5 occurrences of the digit 9 and the first two digits are not the same?

Solution. Consider 3 cases:

1. The first digit is a 9.
   Then the remaining nines have to be in the last 8 slots and the remaining 5 slots can be filled with any of the remaining 9 digits. There are
   \[
   1 \cdot 9 \cdot \binom{8}{4} \cdot 9^4 = \binom{8}{4} \cdot 9^5
   \]
such numbers.

2. The first digit is not a 9, but the second one is.
   Then the remaining nines have to be in the last 8 slots and the remaining 5 slots can again be filled with any of the remaining 9 digits. There are again
   \[
   9 \cdot 1 \cdot \binom{8}{4} \cdot 9^4 = \binom{8}{4} \cdot 9^5
   \]
such numbers.

3. The first two digits are not nines. In this case, there are \( 9 \cdot 8 \) ways to select the first two digits, the 5 nines have to be in the last 8 slots and the remaining 3 slots can be filled with any digit other than 9. This gives
   \[
   9 \cdot 8 \cdot \binom{8}{5} \cdot 9^3
   \]
such numbers.
In the end, the final answer is
\[ 2 \cdot \binom{8}{4} \cdot 9^5 + 8 \cdot \binom{8}{5} \cdot 9^4. \]

Problem 5. [30 points]

(a) [5 pts] How many ways are there to distribute \(n\) dollars among \(k\) people assuming each person has to get an integer amount of dollars?

Solution. We can encode a solution as a string of \(n + k - 1\) bits with \(k - 1\) of these as ones and \(n\) of these as dollars: a zero indicates to give a dollar to the ‘current’ person, and a one indicates to move to the next person. By the bookkeeper rule, the number of such strings is:
\[ \binom{n + k - 1}{k - 1} \]

(b) [5 pts] How about if everyone has to get at least one dollar?

Solution. We can modify the approach of (a) as follows: we are looking for a string consisting of \(n\) tokens ‘0’ or ‘10’, and necessarily starting with 0. This ensures that, if we interpret zeroes and ones as in the solution to part (a), each person gets at least one dollar. Applying the bookkeeper rule to this scenario (excluding the guaranteed leading 0), we obtain:
\[ \binom{n - 1}{k - 1} \]

(c) [10 pts] In how many ways can you arrange \(n\) books on \(k\) bookshelves (assuming the books are distinguishable, so that order matters)?

Solution. There are \(\binom{n+k-1}{k-1}\) ways to distribute \(n\) books to \(k\) bookshelves, if we ignore order, using the same zeroes-and-ones bookkeeper method as in part (a). Given such a distribution, there are now \(n!\) permutations of the particular books. By the (generalized) product rule, the result is:
\[ n! \cdot \binom{n + k - 1}{k - 1} \]

(d) [10 pts] How about if there has to be at least 1 book on each bookshelf?
Solution. Following the approach of part (b), there are \( \binom{n-1}{k-1} \) ways to distribute \( n \) books to \( k \) shelves, ignoring order, if each shelf must have at least one book. Again there are \( n! \) permutations of the particular books in question, so that the total count of arrangements is
\[
n! \binom{n-1}{k-1} = \frac{n!(n-1)!}{(k-1)!(n-k)!}
\]

Problem 6. **[25 points]** We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give \( n \) cents change.

(a) **[4 pts]** Write the generating function \( P(x) \) for the number of ways to use only pennies to make \( n \) cents.

**Solution.** Since there is only one way to change any given amount with only pennies, so the generating function for the sequence is
\[
P(x) := 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.
\]

(b) **[4 pts]** Write the generating function \( N(x) \) for the number of ways to use only nickels to make \( n \) cents.

**Solution.** There is no way to change amounts that are not multiples of five with only nickels. So the generating function for the sequence is
\[
N(x) := 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 1x^5 + 0x^6 + 0x^7 + 0x^8 + 0x^9 + x^{10} + 0x^{11} + \cdots
= 1 + x^5 + x^{10} + x^{15} + \cdots
= \frac{1}{1-x^5}.
\]

(c) **[8 pts]** Write the generating function for the number of ways to use only nickels and pennies to change \( n \) cents.

**Solution.** Since \( P(x) \) and \( N(x) \) are generating functions for the number of ways to choose pennies and nickels separately, by the Convolution Rule for generating functions, the generating function for the number of ways to choose pennies and nickels together, is the their product
\[
N(x) \cdot P(x) = \frac{1}{(1-x)(1-x^5)}.
\]
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(d) [4 pts] Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give \( n \) cents change.

**Solution.** Generalizing our method gives:

\[ C(x) := \frac{1}{(1 - x)(1 - x^5)(1 - x^{10})(1 - x^{25})(1 - x^{50})}. \]

(e) [5 pts] Explain how you could use the function found in part (d) to determine the number of ways to change 50 cents; you do not have to perform your computation or provide its result.

**Solution.** The answer is the coefficient to \( x^{50} \) of the power series for \( C(x) \), which happens to be 50. This coefficient could be extracted by taking the 50th derivative of \( C(x) \). This approach would not be appropriate for hand calculation, but the answer would be easy to get using a symbolic mathematics program such as Mathematica.

Problem 7. [20 points] Recall the operation of taking the derivatives of generating functions. This is done termwise, that is, if

\[ F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots, \]

then

\[ F'(x) := f_1 + 2f_2 x + 3f_3 x^2 + \cdots. \]

For example,

\[ \frac{1}{(1-x)^2} = \left( \frac{1}{1-x} \right)' = 1 + 2x + 3x^2 + \cdots \]

so

\[ H(x) := \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots \]

is the generating function for the sequence of nonnegative integers. Therefore

\[ \frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots, \]

so

\[ \frac{x^2 + x}{(1-x)^3} = xH'(x) = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots + n^2 x^n + \cdots \]

is the generating function for the nonnegative integer squares.

(a) [10 pts] Prove that for all \( k \in \mathbb{N} \), the generating function for the nonnegative integer \( k \)th powers is a quotient of polynomials in \( x \). That is, for all \( k \in \mathbb{N} \) there are polynomials \( R_k(x) \) and \( S_k(x) \) such that

\[ [x^n] \left( \frac{R_k(x)}{S_k(x)} \right) = n^k. \] (2)
(Hint: Observe that if a function $f$ is a quotient of polynomials, then its derivative $f'$ is also a quotient of polynomials. It is not necessary to work out explicit formulas for $R_k$ and $S_k$ to prove this part.)

**Solution.** The proof is by induction on $k$ with the hypothesis that there are polynomials $R_k(x)$ and $S_k(x)$ satisfying (2).

**Base case $k = 0$:** Let $R_0(x) := 1$, $S_0(x) := (1 - x)$.

**Inductive step:** Assuming by induction that we have $R_k, S_k$ satisfying (2), the generating function for the $k + 1$st powers will be

$$x \left( \frac{R_k(x)}{S_k(x)} \right)' = \frac{xR_k'S_k - xR_kS_k'}{S_k^2}$$

But the derivative of a polynomial is a polynomial, so the right hand side of (3) is a sum of quotients of polynomials, which can always be simplified into a quotient of two polynomials. Taking $R_{k+1}$ and $S_{k+1}$ to be these two polynomials, the inductive step is complete. 

(b) [10 pts] Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and $p$ is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in $x$.

(Hint: Consider $R_k(\alpha x)/S_k(\alpha x)$.)

**Solution.** By part (a), $R_k(\alpha x)/S_k(\alpha x)$ is the generating function for the sequence whose $n$th term is $n^k\alpha^n$. So for any function of $n$ of the form $p(n)\alpha^n$, a linear combination of such expressions $R_k/S_k$ for $k = 0, 1, \ldots, \deg(p)$ will yield a quotient, $Q(x)$, of polynomials that is a generating function, for $p(n)\alpha^n$, that is $[x^n]Q(x) = p(n)\alpha^n$.

Letting $F(x)$ be the generating function with $[x^n]F(x) = f(n)$, the usual argument shows that

$$F(x) - axF(x) - bx^2F(x) - cx^3F(x) - Q(x) = rx^2 + sx + t,$$

where

$$r = f(2) - af(1) - bf(0) - p(2)\alpha^2$$
$$s = f(1) - af(0) - p(1)\alpha$$
$$t = f(0) - p(0).$$

This shows that $F(x)$ is a quotient of polynomials, namely,

$$F(x) = \frac{Q(x) + rx^2 + sx + t}{1 - ax - bx^2 - cx^3}. $$
Problem 8. [15 points]

Prove, using generating functions, that

\[
\binom{i + j}{k} = \sum_{\ell=0}^{k} \binom{i}{\ell} \binom{j}{k-\ell}.
\]

(Hint: Think of both expressions as the \(k\)th terms of some sequences whose generating functions you might want to determine. The right-hand side looks like it could be a convolution.)

Solution. The left-hand side is the \(k\)th coefficient of the generating function

\[
\sum_{n=0}^{\infty} \binom{i + j}{n} x^n = (x + 1)^{i+j}.
\]

The right-hand side is the \(k\)th coefficient of the convolution of the following sequences:

\[a_n = \binom{i}{n}, \quad b_n = \binom{j}{n},\]

which have generating functions

\[(x + 1)^i \quad \text{and} \quad (x + 1)^j.\]

The result now follows from the convolution law: the convolution on the right-hand side has generating function

\[(x + 1)^i \cdot (x + 1)^j = (x + 1)^{i+j},\]

which is precisely the generating function for the left-hand side; so the left and right are equal. \(\blacksquare\)