Problem Set 8 Solutions

Due: Monday, November 3

Reading Assignment: Sections 10.3–10.5, 11.1–11.7, optional handout on Akra–Bazzi method posted on course website

Problem 1. [20 points] The Towers of Sheboygan puzzle involves 3 posts and $n$ rings of different sizes. The rings are placed on post #1 in order of size with the smallest ring on top and largest on bottom as in Figure 1.

![Figure 1: The starting position.](image)

The objective is to transfer all $n$ rings to post #2 via a sequence of moves. A move consists of moving the top ring from either post #1 to post #2, from post #2 to post #3, or from post #3 to post #1. For example, moving a ring directly from post #1 to post #3 is not permitted. Moreover, a larger ring can never be placed on top of a smaller ring.

(a) [7 pts] Describe a solution to the Towers of Seboygan puzzle.

Solution. We construct a solution for $n$ disks inductively. As a base case, we are able to move a single disk from Post 1 to Post 2, as one of the given basic moves. Supposing a solution for $n-1$ disks, we move $n$ disks from Post 1 to Post 2 as follows: move the smallest $n-1$ disks from Post 1 to Post 2, then from Post 2 to Post 3 (using the same solution with the roles of posts interchanged); now move the $n$th disk from Post 1 to Post 2; and now move the first $n-1$ disks from Post 3 to Post 1 and onward to Post 2. Now all $n$ disks are on Post 2, as desired.
(b) [6 pts] Let $S_n$ be the number of moves required to solve the $n$-disk problem. Express $S_n$ with a recurrence relation and sufficient base cases.

**Solution.** Let $M(n)$ denote the number of moves required to solve the $n$-disk problem. Our solution to part (a) for $n$ disks invokes the $(n-1)$-disk solution four times, and also makes one additional move, which translates into the following recurrence:

$$M(1) = 1, \quad M(n + 1) = 4M(n) + 1 (n \geq 1).$$

(c) [7 pts] Find a closed-form expression for $S_n$ by solving the recurrence.

**Solution.** Experimentation with the first few terms, or the “plug-and-chug” method, might lead one to guess the following solution:

$$M(n) = \sum_{k=0}^{n-1} 4^k = \frac{1 - 4^n}{1 - 4} = \frac{4^n - 1}{3}.$$

We can prove this by induction. The base case $M(1) = 1$ is clear. Supposing the result for $n$, we have

$$M(n + 1) = 4M(n) + 1 = 4 \cdot \frac{4^n - 1}{3} + 1 = \frac{4^{n+1} - 4 + 3}{3} = \frac{4^{n+1} - 1}{3},$$

completing the induction.

**Problem 2. [15 points]**

Construct a linear recurrence (with sufficient boundary conditions) whose only solution is the following sequence:

$$f(n) = C\alpha^n + D\beta^n,$$

where $\alpha, \beta, C, D$ are real constants.

**Solution.** We can reverse the usual process for solving homogeneous linear recurrences. We want to produce an auxiliary equation with $\alpha$ and $\beta$ as roots, so we take:

$$0 = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

Under our method of solving linear recurrences, this corresponds to the following recurrence:

$$0 = f(n) - (\alpha + \beta)f(n - 1) + \alpha\beta f(n - 2),$$

which we can rewrite as:

$$f(n) = (\alpha + \beta)f(n - 1) - \alpha\beta f(n - 2).$$

In order to determine a unique solution, we should impose two boundary conditions, which we can obtain by evaluating the original sequence at $n = 0$ and $n = 1$:

$$f(0) = C + D, \quad f(1) = \alpha C + \beta D.$$
Problem 3. [25 points] Find $\Theta$ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T([n/2]) + n$

(b) [5 pts] $T(n) = 2T([n/8] + 1/n) + n$

(c) [5 pts] $T(n) = 7T([n/20]) + 2T([n/8]) + n$

(d) [5 pts] $T(n) = 2T([n/4] + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T([n/9 + n^{1/9}]) + 1$

Solution. We use the method of Akra-Bazzi for these problems.

(a) We see that $a = 8, b = 1/2, h = [n/2] - n/2$ so $p = 3$ gives $ab^p = 1$.

$$T(n) = \Theta(n^3(1 + \int_1^n \frac{u}{u^4} du)) = \Theta(n^3(1 + \int_1^n u^{-3} du)) = \Theta(n^3).$$

(b) $a_1 = 2, b_1 = 1/8, h_1(n) = [n/8] - n/8 + 1/n, g(n) = n, p = 1/3,$

$$T(n) = \Theta \left( n^{1/3} \left( 1 + \int_1^n \frac{u}{u^{4/3}} du \right) \right)$$

$$= \Theta \left( n^{1/3} \left( 1 + \int_1^n u^{-1/3} du \right) \right)$$

$$= \Theta \left( n^{1/3} + n^{1/3} \int_1^n u^{-1/3} du \right)$$

$$= \Theta \left( n^{1/3} + n^{1/3} \frac{3}{2} (n^{2/3} - 1) \right)$$

$$= \Theta(n).$$

(c) $a_1 = 7, b_1 = 1/20, a_2 = 2, b_2 = 1/8, h_1(n) = [n/20] - n/20, h_2(n) = [n/8] - n/8, g(n) = n$. Finally, note that although we do not know what $p$ is, we are guaranteed that $p < 1$.

$$T(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) = \Theta(n^p(1 + \int_1^n u^{-p} du))$$

$$= \Theta(n^p + n^p \frac{1}{1-p} (n^{1-p} - 1))$$

$$= \Theta(n).$$

(d) $a_1 = 2, b_1 = 1/4, h_1(n) = [n/4] - n/4 + 1, g(n) = n^{1/2}, p = 1/2,$

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{u^{1/2}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$
(e) \(a_1 = 3, b_1 = 1/9, h_1(n) = \lfloor n/9 + n^{1/9} \rfloor - n/9, g(n) = 1, p = 1/2,\)
\[T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{1}{u^{3/2}} du)) = \Theta(n^{1/2}).\]

**Problem 4. [30 points]** Find closed-form solutions to the following linear recurrences.

(a) [15 pts] \(x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)\)

**Solution.** The characteristic equation is \(r^3 - 4r^2 + r + 6 = 0.\)
Generally, solving a cubic equation is a difficult problem. However, we can find from inspection that the roots are:
\[r_1 = -1, \quad r_2 = 2, \quad r_3 = 3\]
Therefore a general form for a solution is
\[x_n = A(-1)^n + B(2)^n + C(3)^n.\]
Substituting the initial conditions into this general form gives a system of linear equations.
\[
\begin{align*}
3 &= A + B + C \\
4 &= -A + 2B + 3C \\
14 &= A + 4B + 9C
\end{align*}
\]
The solution to this linear system is \(A = 1, B = 1,\) and \(C = 1.\) The complete solution to the recurrence is therefore
\[x_n = (-1)^n + 2^n + 3^n.\]

(b) [15 pts] \(x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)\)

**Solution.** First, we find the general solution to the homogenous recurrence. The characteristic equation is \(r^2 + r - 2 = 0.\) The roots of this equation are \(r_1 = 1\) and \(r_2 = -2.\) Therefore, the general solution to the homogenous recurrence is
\[x_n = A(-1)^n + B2^n.\]
Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess there is a linear solution, that is, a solution of the form $an + b$. If the solution is of this form, we must have

$$an + b = -a(n - 1) - b + 2a(n - 2) + 2b + n$$

Gathering up like terms, this simplifies to

$$n(a + a - 2a - 1) + (b + a + b + 4a - 2b) = 0$$

which implies that

$$n = -5a$$

But $a$ is a constant, so this cannot be so. So we make another guess, this time that there is a quadratic solution of the form $an^2 + bn + c$. If the solution is of this form, we must have

$$an^2 + bn + c = -[a(n - 1)^2 + b(n - 1) + c] + 2[a(n - 2)^2 + b(n - 2) + c] + n$$

which simplifies to

$$n^2(a + a - 2a - 1) + n(b + b - 2a + 8a - 2b - 1) + (c + a - b + c - 8a + 4b - 2c) = 0$$

This simplifies to

$$n(6a - 1) + (-7a + 3b) = 0$$

This last equation is satisfied only if the coefficient of $n$ and the constant term are both zero, which implies $a = 1/6$ and $b = 7/18$. Apparently, any value of $c$ gives a valid particular solution. For simplicity, we choose $c = 0$ and obtain the particular solution:

$$x_n = \frac{1}{6}n^2 - \frac{7}{18}n.$$

The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A(-1)^n + B2^n + \frac{1}{6}n^2 - \frac{7}{18}n$$

Substituting the initial conditions gives a system of linear equations:

$$5 = A + B$$
$$-4/9 = -A + 2B - +1/6 + 7/18$$

The solution to this linear system is $A = 3$ and $B = 2$. Therefore, the complete solution to the recurrence is

$$x_n = 3 + 2(-2)^n + \frac{1}{6}n^2 + \frac{7}{18}n$$
Problem 5. [50 points] Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [5 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

**Solution.** There is a bijection with sequence of the form:

(value of pair, suits of pair, value of other three cards, suits of other three cards)

Thus, the number of hands with a single pair is:

\[ 13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240 \]

Alternatively, there is also a 3!-to-1 mapping to the tuple:

(value of pair, suits of pair, value 3rd card, suit 3rd card, value 4th card, suit 4th card, value 5th card, suit 5th card)

Thus, the number of hands with a single pair is:

\[ \frac{13 \cdot \binom{4}{2} \cdot 12 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 4}{3!} = 1,098,240 \]

(b) [5 pts] How many 5-card hands have two or more kings?

**Solution.** This is the set of all hands minus the hands with either no kings or one king:

\[ \binom{52}{5} - \binom{48}{5} - 4 \cdot \binom{48}{4} = 108,336 \]

Alternatively, this is also the set of all hands of two, three, or four kings:

\[ \binom{48}{3} \cdot \binom{4}{2} + \binom{48}{2} \cdot \binom{4}{3} + \binom{48}{1} \cdot \binom{4}{4} = 108,336 \]

(c) [5 pts] How many 5-card hands contain the ace of spades, the ace of clubs, or both?
**Solution.** There are $\binom{51}{4}$ hands containing the ace of spades, an equal number containing the ace of clubs and $\binom{50}{3}$ containing both. By the inclusion-exclusion formula:

$$\binom{51}{4} + \binom{51}{4} - \binom{50}{3}$$

hands contain one or the other or both. ■

(d) [5 pts] For fixed positive integers $n$ and $k$, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i = n$$

**Solution.** There is a bijection from the solutions of the equation to the binary strings containing $n$ zeros and $k$ ones where $x_0$ is the number of 0s preceding the first 1, $x_k$ is the number of 0s following the last 1 and $x_i$ is the number of 0s between the $i^{th}$ and $(i+1)^{th}$ 1 for $0 < i < k$.

$$\binom{n+k}{k}$$

■

(e) [5 pts] For fixed positive integers $n$ and $k$, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i \leq n$$

**Solution.** There is a bijection from the solutions of

$$\sum_{i=0}^{k} x_i \leq n$$

$$= n - x_{k+1} \quad \text{(for some } x_{k+1} \geq 0)$$

and the solutions of

$$\sum_{i=0}^{k+1} x_i = n.$$
(f) [5 pts] In how many ways can $3n$ students be broken up into $n$ groups of 3?

Solution. 
\[
\frac{(3n)!}{(3!)^n n!}.
\]

(g) [5 pts] How many simple undirected graphs are there with $n$ vertices?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:
\[
2^{\binom{n}{2}}.
\]

(h) [5 pts] How many directed graphs are there with $n$ vertices (self loops allowed)?

Solution. There are $n^2$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:
\[
2^{n^2}.
\]

(i) [5 pts] How many tournament graphs are there with $n$ vertices?

Solution. There are no self-loops in a tournament graph and for each of the $\binom{n}{2}$ pairs of distinct vertices $a$ and $b$, either $a \to b$ or $b \to a$ but not both. Therefore, the number of tournament graphs is:
\[
2^{\binom{n}{2}}.
\]

(j) [5 pts] How many acyclic tournament graphs are there with $n$ vertices?

Solution. For any path from $x$ to $y$ in a tournament graph, an edge $y \to x$ would create a cycle. Therefore in any acyclic tournament graph, the existence of a path between distinct vertices $x$ and $y$ would require the edge $x \to y$ also be in the graph. That is, the ”beats” relation for such a graph would be transitive. Since each pair of distinct players are comparable (either $x \to y$ or $y \to x$) we can uniquely rank the players $x_1 < x_2 < \cdots < x_n$. There are $n!$ such rankings.

Problem 6. [10 points] Suppose we have a deck of cards that has 4 suits, each suit having 13 cards. The magician asks the audience to select an arbitrary set of 7 cards. His assistant selects 4 out of the 7 cards and puts these 4 cards on a table (in a chosen order). If the magician and assistant pre-arrange a strategy, is it possible that the magician can figure out the identities of the 3 remaining cards that are hidden from him, by only considering the 4 cards that his assistant put on the table?
Solution. No. The assistant has $\binom{7}{4}$ choices to select 4 cards and 4! choices to order these cards. This gives $\binom{7}{4}4! = 7!/3! = 840$ ways to map a set of 7 cards to a sequence of 4 cards.

There are 3 hidden cards. The complete deck has 52 cards and 4 cards are visible. So, the three hidden cards are from a set of 48 cards. There are $\binom{48}{3} = 48 \cdot 47 \cdot 46/3! = 17296$ ways to select 3 out of 52 cards.

Since the number of possible mappings is less than the number of possible triples of hidden cards, it is not possible for the assistant and magician to agree on a mapping such that the magician is able to find the identities of the hidden cards. ■

Problem 7. [15 points]

In this problem, we will discuss the lower bound for the number of comparisons an algorithm needs to make in order to sort $n$ numbers. We can represent this problem with the following scenario: suppose that an opponent has $n$ boxes lined up in a row, and in each of the $n$ boxes is a number (no range specified, though you may assume that the numbers are distinct). Your goal is to figure out which box has the smallest number, which box has the next smallest number, and so on, up to the box containing the largest number; that is, you need to figure out the particular order in which the boxes have been arranged. You win the scenario if you are able to determine the order after at most $k$ questions. However, the only question you can ask to gather information about the order of the boxes is to select two boxes and ask which one has the bigger number.

(a) [2 pts] Explain why this problem is equivalent to the problem where the numbers in the boxes are simply 1 through $n$.

Solution. If we re-number the lowest-numbered box as 1, the second-lowest numbered box as 2, and so on up to $n$, this doesn’t change the order. ■

(b) [1 pts] How many permutations (i.e. distinct arrangements) of these boxes are there?

Solution. The number of permutations of $n$ items is $n!$. ■

In principle, you can use the idea of a decision tree to plan the questions you will ask in order to figure out the permutation in which the boxes happen to be arranged (i.e. the order). You would first compare two boxes, and then depending on the outcome of the first comparison, you would compare some other pair of boxes (a different pair, perhaps, depending on the outcome of the first two boxes). Now, we don’t know which pair of boxes you should compare first, or what the next two potential pairs of boxes would be; we’re just saying it could be done in principle!

(c) [5 pts] Explain why you cannot guarantee a win if $2^k < n!$. (Hint: Think about how many permutations you can differentiate with each question you ask.)

Solution. We show by induction that, after asking $r$ questions, it is impossible to guarantee that there are fewer than $\frac{n!}{2^r}$ permutations satisfying the answers to those questions. The base case is part (b): there are $n!$ permutations in total.
For the inductive step, assume that \( r \) questions have been asked; by inductive hypothesis, in the worst case there are still at least \( \frac{n!}{2^r} \) permutations satisfying the answers to these questions. We ask our next comparison question, comparing two boxes; as there are only two possible outcomes to this question, one of those outcomes must be correct for at least half of the remaining permutations, i.e. for at least \( \frac{n!}{2^r} \) permutations. In the worst case, this is the answer that we receive to our question; this completes the inductive hypothesis.

In particular, for \( r = k \), there remain at least \( \frac{n!}{2^k} \) permutations that satisfy all questions asked; so if \( n! > n^k \), there is more than one remaining permutation, and we have failed to determine the correct permutation.

\[ \square \]

(d) [4 pts] Show that there exists a constant \( c > 0 \) such that if you are only allowed to ask at most \( k = c \cdot n \log n \) questions, you cannot guarantee a win.

Solution. We have shown that it is impossible to guarantee a win if \( 2^k < n! \), i.e. if \( k < \log_2(n!) \). But \( \log_2(n!) = O(n \log n) \), by Stirling’s approximation:

\[
\log_2(n!) \sim \log_2(\sqrt{2\pi n}) + \frac{1}{2} \log_2(n) + n(\log_2 n - \log_2 e) = O(n \log n).
\]

To paraphrase big-\( O \) notation, there is some constant \( c > 0 \) such that, for sufficiently large \( n \), it is impossible to guarantee a win by asking only \( k = c \cdot n \log n \) questions.

\[ \square \]

(e) [3 pts] Deduce from this that there is no sorting algorithm (based only on comparing elements) that can sort any \( n \)-element array with \( o(n \log n) \) comparisons.

Solution. By contradiction. Supposing that we had such a sorting algorithm; then we could apply it to our \( n \) boxes in the problem above, and then read off the order of the boxes from the sorted list. So we could determine the order of the \( n \) boxes using \( o(n \log n) \) comparisons. By definition of \( o \)-notation, for all sufficiently large \( n \), we determine the order of the boxes in fewer than \( c \cdot n \log n \) comparisons, where \( c \) is the constant from part (d). This contradicts the result of part (d), and the result follows.

\[ \square \]