Problem Set 5

Due: Monday, October 6

Reading Assignment: Sections 5.4.1–5.4.2, 5.5–5.7, 6.1–6.3

Problem 1. [16 points]
In the cycle $C_{2n}$ of length $2n$, we'll call two vertices opposite if they are on opposite sides of the cycle, that is that are distance $n$ apart in $C_n$. Let $G$ be the graph formed from $C_{2n}$ by adding an edge, which we'll call a crossing edge, between each pair of opposite vertices. So $G$ has $n$ crossing edges.

(a) [6 pts] Give a simple description of the shortest path between any two vertices of $G$. 

*Hint: Argue that a shortest path between two vertices in $G$ uses at most one crossing edge.*

(b) [3 pts] What is the diameter of $G$, that is, the largest distance between two vertices?

(c) [3 pts] We say that a graph is $k$-edge connected if removing $(k - 1)$ edges can not disconnect the graph. Prove that the graph above is not 4-edge connected.

(d) [4 pts] Prove that the graph is 3-edge connected.

Problem 2. [14 points]
Bruce and Sam have been told that there is a bomb on a street intersection, lying in a region of Manhattan for which the street map forms a $19 \times 7$ undirected grid. (Vertices are street intersections, and edges are single blocks of a street). The bomb is on one street intersection, and the code that they need to defuse the bomb is on another street intersection. Starting from where the bomb is, Bruce and Sam need to check all $19 \cdot 7 = 133$ street intersections for the code. Once they are at an intersection, they don’t need any additional time to verify whether the code is there. Once they find the code and return to the bomb, they can disarm it in 2 minutes. Also, they can run one block (in any of the four directions) in exactly 1 minute. They are given 135 minutes total to find the code and disarm the bomb, which means that they need to return to the bomb, code in hand, in 133 minutes.

Sam realizes that they need to use a cool new 6.042 concept: a Hamiltonian cycle is a path that visits each vertex in a graph exactly once and ends at its starting point (so it is a cycle). A graph is Hamiltonian if it has an Hamiltonian cycle. Sam is very excited because he thinks he can show that this region of Manhattan is Hamiltonian. If it is, Bruce and Sam can save the day! Will they make it?
(a) [5 pts] Show that they cannot do it – that is, more generally, show that if both $N$ and $M$ are odd, then the $N \times M$ grid is not Hamiltonian. *Hint: First show that any $N \times M$ 2-dimensional undirected grid is bipartite.*

(b) [9 pts] Suppose we defined Midtown in the more standard way as extending from 40th Street to 59th Street and from 3rd Avenue to 9th Avenue (a $20 \times 7$ grid), and gave them another 7 minutes.

1. Show that if either $N$ is even and $M > 1$ or $M$ is even and $N > 1$, then the $N \times M$ grid is Hamiltonian.
2. Explain why your proof breaks down when $N$ and $M$ are odd.
3. Would they survive? Does the outcome depend on where the bomb is placed?

**Problem 3. [16 points]**

(a) [8 pts] Prove that a simple connected graph with $n$ nodes and $n - 1$ edges is a tree.

(b) [8 pts] Prove by induction that any connected graph has a spanning tree.

**Problem 4. [13 points]**

Let $G$ be a weighted undirected graph, and assume that $G$ is connected.

(a) [8 pts] Suppose that $G$ contains a cycle $C$, and that we produce a subgraph $G'$ of $G$ by deleting a maximum-weight edge $e$ from $C$. Show that any minimum-weight spanning tree (MST) for $G'$ is an MST for $G$.

(b) [5 pts] Suppose that we iterate the process of part (a): while $G$ has a cycle $C$, find a highest-weight edge along $C$ and delete it. Prove that this procedure terminates in a minimum-weight spanning tree for $G$.

**Problem 5. [10 points]**

Show that the congestion of the $N$-input butterfly is $\sqrt{N}$ if $N$ is an even power of 2.

**Problem 6. [20 points]**

In a *perfect shuffle*, a deck of $N$ cards is cut exactly in half and then perfectly interlaced. Thus, for $N$ cards, we would obtain the resulting cards in the following order:

$$1, \left(\frac{N}{2} + 1\right), 2, \left(\frac{N}{2} + 2\right), \ldots, \left(\frac{N}{2} - 1\right), (N - 1), \frac{N}{2}, N.$$ 

(a) [10 pts] Show that $m$ perfect shuffles will return a deck of $N$ cards to its original order provided that $2^m = 1 \pmod{(N - 1)}$. 
(b) [4 pts] Show that 8 perfect shuffles are necessary and sufficient to return a deck of 52 cards to their original order.

(c) [5 pts] How many perfect shuffles are necessary and sufficient to return a deck of cards to its original order if there are two jokers added to the deck (so that it has 54 cards)?

Problem 7. [8 points] For the Grid, Interrupted switching network (Figure 1), find the diameter and congestion. Give a reason for your answer.

![Grid, Interrupted](image_url)

Figure 1: Grid, Interrupted.

Problem 8. [10 points] Construct a 16-bit de Bruijn sequence, by considering an Eulerian tour of the de Bruijn graph.