Staff Solutions to Problem Set 9

Reading: Section 13.7. Asymptotics; Chapter 14. Cardinality Rules through 14.2. Counting with Bijections

STAFF NOTE: Lectures covered: Asymptotics, Counting with Bijections

Problem 1.
Let \( f, g \) be nonnegative real-valued functions such that \( \lim_{x \to \infty} f(x) = \infty \) and \( f \sim g \).

(a) Give an example of \( f, g \) such that \( \not{\forall} (2^f \sim 2^g) \).

Solution.

\[
\begin{align*}
    f(n) &:= n + 1 \\
    g(n) &:= n.
\end{align*}
\]

Then \( f \sim g \) since \( \lim [(n + 1)/n] = 1 \), but \( 2^f = 2^{n+1} = 2 \cdot 2^n = 2^g \) so

\[
\lim \frac{2^f}{2^g} = 2 \neq 1.
\]

(b) Prove that \( \log f \sim \log g \).

Solution.

\[
\begin{align*}
    \lim \frac{f}{g} &= 1 \\
    \log \lim \frac{f}{g} &= \log 1 \\
    \lim \log \frac{f}{g} &= 0 & \text{since log is continuous on } \mathbb{R}^+ \\
    \lim (\log f - \log g) &= 0 \\
    \lim \frac{\log f}{\log g} &= 0 \\
    \lim \frac{\log f - \log g}{\log g} &= 0 \\
    \lim \frac{\log f}{\log g} - 1 &= 0 \\
    \lim \frac{\log f}{\log g} &= 1
\end{align*}
\]

Note that this proof did not need the condition that \( \lim_{x \to \infty} f(x) = \infty \).
(c) Use Stirling’s formula to prove that in fact
\[
\log(n!) \sim n \log n
\]

**Solution.** Taking logs of both sides of Stirling’s formula, we have
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \text{(Stirling)}
\]
\[
\log(n!) \sim n \log \left(\frac{n}{e}\right) + \log \sqrt{2\pi n}
\]
\[
= n \log n - n \log e + \log \sqrt{2\pi n}
\]
\[
\sim n \log n.
\]

The final step follows from the fact that
\[
\lim_{n \to \infty} \frac{n \log n - n \log e + \log \sqrt{2\pi n}}{n \log n} = 1 - \lim_{n \to \infty} \frac{\log e}{\log n} + \lim_{n \to \infty} \frac{\log \sqrt{2\pi n}}{n \log n} + \lim_{n \to \infty} \frac{1}{2n}
\]
\[
= 1 - 0 - 0 - 0 = 1.
\]

**Problem 2.**  (a) Either prove or disprove each of the following statements.

- \( n! = O((n + 1)! \)
- \( (n + 1)! = O(n!) \)
- \( n! = \Theta((n + 1)! \)
- \( n! = o((n + 1)! \)
- \( (n + 1)! = o(n!) \)

**Solution.** Observe that:
\[
\lim_{n \to \infty} \frac{n!}{(n + 1)!} = \lim_{n \to \infty} 1/(n + 1) = 0.
\]
This gives us \( n! = o((n + 1)! \) as well as \( n! = O((n + 1)! \).

Further,
\[
\lim_{n \to \infty} \frac{(n + 1)!}{n!} = \lim_{n \to \infty} n + 1 = \infty.
\]
This implies that \( (n + 1)! = O(n!) \) and \( (n + 1)! = o(n!) \) are false. Combining the two limits, we get that \( n! = \Theta((n + 1)! \) must also be false.

(b) Show that \( \left(\frac{n}{3}\right)^{n+c} = o(n!) \).

**Solution.** By Stirling’s formula,
\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]
On the other hand, note that \( \left(\frac{n}{3}\right)^{n+c} = \left(\frac{n}{3}\right)^c \left(\frac{n}{3}\right)^n \). Dividing this quantity by \( n! \), we get:
\[
\frac{\left(\frac{n}{3}\right)^c \left(\frac{n}{3}\right)^n}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = \frac{n^{c-1/2}}{3^c \sqrt{2\pi}} \left(\frac{e}{3}\right)^n.
\]
This expression goes to 0 in the limit as \( e < 3 \). Thus, \( \left( \frac{2}{3} \right)^{n+e} = o(n!) \).

Problem 3.
Suppose you have seven dice—each a different color of the rainbow; otherwise the dice are standard, with faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is \((3, 1, 6, 1, 4, 5, 2)\) indicating that the red die showed a 3, the orange die showed 1, the yellow 6, . . . .

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple arithmetic formula, possibly involving factorials and binomial coefficients, for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let \( A \) be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let \( R \) be the set of seven rainbow colors and \( S := [1, 6] \) be the set of dice values.

Define \( B := P_{S,2} \times R_3 \), where \( P_{S,2} \) is the set of 2-permutations of \( S \) and \( R_3 \) is the set of size-3 subsets of \( R \). Then define a bijection from \( A \) to \( B \) by mapping a roll in \( A \) to the sequence in \( B \) whose first element is an ordered pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll

\[
(4, 4, 2, 2, 4, 2, 4) \in A
\]

maps to

\[
((2, 4), \{\text{yellow, green, indigo}\}) \in B.
\]

Now by the Bijection rule \(|A| = |B|\), and by the Generalized Product and Subset rules,

\[
|B| = 6 \cdot 5 \cdot \binom{7}{2}.
\]

(a) For how many rolls do *exactly* two dice have the value 6 and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: \((6, 2, 6, 1, 3, 4, 5)\) is a roll of this type, but \((1, 1, 2, 6, 3, 4, 5)\) and \((6, 6, 1, 2, 4, 3, 4)\) are not.

Solution. As in the example, map a roll into an element of \( B := R_2 \times P_5 \) where \( P_5 \) is the set of permutations of \( \{1, \ldots, 5\} \): A roll maps to the pair whose first element is the set of colors of the two dice with value 6, and whose second element is the sequence of values of the remaining dice (in rainbow order). So \((6, 2, 6, 1, 3, 4, 5)\) above maps to \((\{\text{red, yellow}\}, (2, 1, 3, 4, 5))\). By the Product rule,

\[
|B| = \binom{7}{2} \cdot 5!.
\]

(b) For how many rolls do two dice have the same value and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: \((4, 2, 4, 1, 3, 6, 5)\) is a roll of this type, but \((1, 1, 2, 6, 1, 4, 5)\) and \((6, 6, 1, 2, 4, 3, 4)\) are not.
Solution. Map a roll into a triple whose first element is in $S$, indicating the value of the pair of matching dice, whose second element is the set of colors of the two matching dice, and whose third element is the sequence of the remaining five dice values (in rainbow order).

So $(4, 2, 4, 1, 3, 6, 5)$ above maps to $(4, \{\text{red,yellow}\}, (2, 1, 3, 6, 5))$. Notice that the number of choices for the third element of a triple is the number of permutations of the remaining five values, namely $5!$. This mapping is a bijection, so the number of such rolls equals the number of such triples. By the Generalized Product rule, the number of such triples is

$$6 \cdot \binom{7}{2} \cdot 5!.$$

Alternatively, we can define a map from rolls in this part to the rolls in part (a), by replacing the value of the duplicated values with 6’s and replacing any 6 in the remaining values by the value of the duplicated pair. So the roll $(4, 2, 4, 1, 3, 6, 5)$ would map to the roll $(6, 2, 6, 1, 3, 4, 5)$. Now a type $a$ roll, $r$, is mapped to by exactly the rolls obtainable from $r$ by exchanging occurrences of 6’s and i’s, for $i = 1, \ldots, 6$. So this map is 6-to-1, and by the Division rule, the number of rolls here is 6 times the number of rolls in part (a).

(c) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value? Remember to describe a bijection and write a simple arithmetic formula.

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Solution. Map a roll of this kind into a 4-tuple whose first element is the set of two numbers of the two pairs of matching dice, whose second element is the set of two colors of the pair of matching dice with the smaller number, whose third element is the set of two colors of the larger of the matching pairs, and whose fourth element is the value of the remaining three dice. For example, the roll $(6, 1, 2, 1, 2, 6, 6)$ maps to the triple

$$(\{1, 2\}, \{\text{orange,green}\}, \{\text{yellow,blue}\}, 6).$$

There are $\binom{6}{2}$ possible first elements of a triple, $\binom{7}{2}$ second elements, $\binom{5}{2}$ third elements since the second set of two colors must be different from the first two, and 4 ways to choose the value of the three dice since their value must differ from the values of the two pairs. So by the Generalized Product rule, there are

$$\binom{6}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot 4$$

possible rolls of this kind.