Staff Solutions to Mini-Quiz 11-4

STAFF NOTE: tree, sums, asymptotics

Problem 1 (Big Oh) (3 points).
Show that
\[ \ln \left( (n^2)! \right) = O \left( n^2 \ln(n) \right). \]

Solution.
\[
\begin{align*}
\ln \left( (n^2)! \right) &= \ln \left( \prod_{i=1}^{n^2} i \right) \\
&= \sum_{i=1}^{n^2} \ln i \\
&\leq \sum_{i=1}^{n^2} 1 \ln n^2 \\
&= n^2 \cdot 2 \ln n \\
&= O(n^2 \ln n).
\end{align*}
\]

Problem 2 (Asymptotic Partial Orders) (4 points).

(a) Indicate which of the following asymptotic relations on the set of positive real-valued functions are equivalence relations, (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

- \( f = o(g) \), the “little Oh” relation.
  
  Solution. S

- \( f = O(g) \), the “big Oh” relation.
  
  Solution. N because it is neither symmetric nor antisymmetric.

- \( f = \Theta(g) \), the “Theta” relation.
  
  Solution. E

- \( f \sim g \), the “asymptotically equal” relation.
  
  Solution. E
• \( f = g \text{ OR } [f = O(g) \text{ AND NOT}(g = O(f))] \).

**Solution.** W.  

(b) Indicate the implications among the assertions in part (a)

**Solution.**

\[
f \sim g \text{ IMPLIES } f = \Theta(g) \text{ IMPLIES } f = O(g),
f = o(g) \text{ IMPLIES } f = O(g) \text{ AND NOT}(g = O(f)) \text{ IMPLIES } f = g \text{ OR } [f = O(g) \text{ AND NOT}(g = O(f))] \text{ IMPLIES }
\]

**Problem 3 (Spanning Trees) (3 points).**
Suppose \( G \) is a connected simple graph with \( n \) vertices and \( 2n - 3 \) edges. Prove that it is impossible to find two spanning trees of \( G \) that do not share some edge.

**Solution.** Every spanning tree of \( G \) must have \( n - 1 \) edges. If two such trees did not share an edge, then there would be at least \( 2n - 2 \) edges, contradicting the fact that there are only \( 2n - 3 \) of them.