Staff Solutions to In-Class Problems Week 15, Wed.

Problem 1. (a) Find a stationary distribution for the random walk graph in Figure 1.

Solution. \[ d(x) = d(y) = \frac{1}{2} \]

Figure 1

(b) If you start at node \( x \) in Figure 1 and take a (long) random walk, does the distribution over nodes ever get close to the stationary distribution? Explain.

Solution. No! You just alternate between nodes \( x \) and \( y \).

In general, a starting distribution \( (p, 1 - p) \) will oscillate between itself and \( (1 - p, p) \), so it will converge to a single distribution if \( p = \frac{1}{2} \). That is, the stationary distribution is the only initial distribution that converges to a stationary distribution.

(c) Find a stationary distribution for the random walk graph in Figure 2.

Solution. \[ d(w) = \frac{9}{19}, \ d(z) = \frac{10}{19} \]. You can derive this by setting \( d(w) = \frac{9}{10}d(z), \ d(z) = d(w) + \frac{1}{10}d(z), \) and \( d(w) + d(z) = 1 \). There is a unique solution.

(d) If you start at node \( w \) Figure 2 and take a (long) random walk, does the distribution over nodes ever get close to the stationary distribution? You needn’t prove anything here, just write out a few steps and see what’s happening.
Solution. Yes, it does. The graph in Figure 2 is strongly connected, and as mentioned in Section ?? and proved in Problem 20.13, strongly connected graphs have unique stationary distributions.

(e) Find a stationary distribution for the random walk graph in Figure 3.

![Figure 3](image)

Solution. There are uncountably many: any distribution with \( d(b) = d(c) = 0 \), and \( d(a) = p \) and \( d(d) = 1 - p \) is stable for any real number \( p \) in the unit interval \([0, 1] \).

(f) If you start at node \( b \) in Figure 3 and take a long random walk, the probability you are at node \( d \) will be close to what fraction? Explain.

Solution. 1/3.

An easy, alternative argument uses the symmetry of the graph. Let \( x \) be the probability that starting at node \( b \) you wind up stuck in \( d \). The only way to get to \( d \) from \( b \) is through \( c \), so the probability of getting stuck at \( d \) starting from \( b \) is the probability of moving to \( c \), namely \( 1/2 \), times the probability of getting stuck at \( d \) starting at \( c \). Now by symmetry, the probability that starting at node \( c \) you wind up stuck in \( a \) is also \( x \). Since the probability of never getting stuck at either \( a \) or \( d \) is obviously 0, the probability that starting at node \( c \) you wind up stuck at \( d \) must be \( 1 - x \). Therefore,

\[
x = \frac{1}{2} (1 - x),
\]

and so \( x = 1/3 \).

Problem 2.

A Google-graph is a random-walk graph such that every edge leaving any given vertex has the same probability. That is, the probability of each edge \( (v \rightarrow w) \) is \( 1/\text{outdeg}(v) \).

A digraph is symmetric if, whenever \( (v \rightarrow w) \) is an edge, so is \( (w \rightarrow v) \). Given any finite, symmetric Google-graph, let

\[
d(v) := \frac{\text{outdeg}(v)}{e},
\]

where \( e \) is the total number of edges in the graph.

(a) If \( d \) was used for webpage ranking, how could you hack this to give your page a high rank? ...and explain informally why this wouldn’t work for “real” page rank using digraphs?

Solution. Just fabricate a large number of new pages and link them all to your page.

This wouldn’t work in the directed case because the fabricated pages would not have “real” links pointing to them, and so would have little if any weight to contribute to the weight of your page.
(b) Show that \( d \) is a stationary distribution.

**Solution.** To show that \( d \) is a stationary distribution, we must show that

\[
d(w) = \sum_{v \in \text{into}(w)} p(v, w) d(v),
\]

where \( \text{into}(w) \) := \( \{ v \mid (v \rightarrow w) \text{ is an edge} \} \).

We have

\[
\sum_{v \in \text{into}(w)} p(v, w) d(v)
= \sum_{v \in \text{into}(w)} \left( \frac{1}{\text{outdeg}(v)} \right) \left( \frac{\text{outdeg}(v)}{e} \right)
= \sum_{v \in \text{into}(w)} \frac{1}{e} \frac{\text{into}(w)}{e} = \frac{\text{indeg}(w)}{e}
= \frac{\text{outdeg}(w)}{e}
= d(w)
\]

(by symmetry of the graph)

(def of \( d \)).