Staff Solutions to In-Class Problems Week 13, Wed.

STAFF NOTE: Expectation, Ch. 18.4-18.5

Problem 1.
Here’s a dice game with payoff parameter $k$: make three independent rolls of a fair die. If you roll a six
- no times, then you lose 1 dollar.
- exactly once, then you win 1 dollar.
- exactly twice, then you win two dollars.
- all three times, then you win $k$ dollars.

For what value of $k$ is this game fair?\(^1\)

Solution. Let the random variable $P$ be your payoff. Then we can compute $\text{Ex}[P]$ as follows:

\[
\text{Ex}[P] = -1 \cdot \Pr[0 \text{ sixes}] + 1 \cdot \Pr[1 \text{ six}] + 2 \cdot \Pr[2 \text{ sixes}] + k \cdot \Pr[3 \text{ sixes}]
\]

\[
= -1 \cdot \left( \frac{5}{6} \right)^3 + 1 \cdot 3 \cdot \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^2 + 2 \cdot 3 \cdot \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right) + k \cdot \left( \frac{1}{6} \right)^3
\]

\[
= \frac{-125 + 75 + 30 + k}{216}
\]

The game is fair when $\text{Ex}[P] = 0$. This happens when $k = 20$.

Problem 2.
A classroom has sixteen desks in a $4 \times 4$ arrangement as shown below.

\[\text{Creative Commons 2013, Eric Lehman, F Tom Leighton, Albert R Meyer .}\]
\[\text{\(^1\)This game is actually offered in casinos with } k = 3, \text{ where it is called Carnival Dice.} \]
If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them flirt. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples? Hint: Linearity.

**Solution.** A natural first approach to this problem is to calculate the expected number of flirtations that each desk is involved in, add the expectations for each desk, and then divide by two (since each flirtation involves to desks). This works fine, but requires finding expectations for three different kinds of desks: corner, mid-side and middle.

A more elegant approach is to note that the expected number of flirtations between adjacent desks is 1/2, and the number of pairs of adjacent desks is 24—there are 12 pairs adjacent horizontally (3 in each of 4 rows) and likewise 12 pairs adjacent vertically. So by linearity of expectation, the expected number of flirtations is \( (1/2) \cdot 24 = 12 \).

To be more explicit about this application of linearity, let’s arbitrarily number the pairs of adjacent desks from 1 to 24 and let \( F_i \) be an indicator for the event that occupants of the desks in the \( i \)-th pair are flirting. The occupants of adjacent desks are flirting iff they are of opposite sexes, which happens with probability 1/2. That is, \( \Pr[F_i = 1] = 1/2 \). The expectation of an indicator variable is the same as the probability it equals 1, so

\[
\text{Ex}[F_i] = 1/2. \tag{1}
\]

Now if \( F \) is the number of flirting couples, then \( F = \sum_{i=1}^{24} F_i \), so the expectation we want is:

\[
\text{Ex}[F] = \text{Ex} \left( \sum_{i=1}^{24} F_i \right) \\
= \sum_{i=1}^{24} \text{Ex}[F_i] \quad \text{(linearity of Ex[\cdot])} \\
= \sum_{i=1}^{24} 1/2 \quad \text{(equation (1))} \\
= 24 \cdot (1/2) = 12.
\]

**Problem 3. (a)** Suppose we flip a fair coin and let \( N_{TT} \) be the number of flips until the first time two Tails in a row appear. What is \( \text{Ex}[N_{TT}] \)?

**Hint:** Let \( D \) be the tree diagram for this process. Explain why \( D \) can be described by the tree in Figure 1. Use the **Law of Total Expectation:** Let \( R \) be a random variable and \( A_1, A_2, \ldots \) be a partition of the sample space. Then

\[
\text{Ex}[R] = \sum_i \text{Ex}[R \mid A_i] \Pr[A_i].
\]

**STAFF NOTE:** Ask what’s wrong with the following argument: the *probability of flipping two heads in a row in 1/4, so by the mean time to failure rule, \( \text{Ex}[N_{TT}] = 1/(1/4) = 4 \), contradicting the value 6 derived above.

The answer is that MTF applies only if the probability of failure at each step is the same independent of the previous flips, but that is obviously not true here, since the probability is twice as large if the previous flip was \( \text{T} \) than if it was \( \text{H} \).
Figure 1 Sample space tree for coin toss until two consecutive heads.

Solution.

\[ \text{Ex}[N_{TT}] = 6. \]

Let \( H \) be the event that a Head appears on the first flip, \( TH \) the event that the first flips are Tail then Head, and likewise \( TT \). From \( D \) and Total Expectation:

\[
\begin{align*}
\text{Ex}[N_{TT}] &= \text{Ex}[N_{TT} \mid H] \cdot \text{Pr}[H] + \text{Ex}[N_{TT} \mid TH] \cdot \text{Pr}[TH] + \text{Ex}[N_{TT} \mid TT] \cdot \text{Pr}[TT] \\
&= (1 + \text{Ex}[N_{TT}]) \cdot \frac{1}{2} + (2 + \text{Ex}[N_{TT}]) \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} \\
&= \frac{1}{2} + \frac{\text{Ex}[N_{TT}]}{2} + \frac{1}{2} + \frac{\text{Ex}[N_{TT}]}{4} + \frac{1}{2} \\
&= \frac{3}{2} + \frac{3 \text{Ex}[N_{TT}]}{4} \\
&\Rightarrow \frac{\text{Ex}[N_{TT}]}{4} = \frac{3}{2}.
\end{align*}
\]

So

\[ \frac{\text{Ex}[N_{TT}]}{4} = \frac{3}{2}. \]

(b) Suppose we flip a fair coin until a Tail immediately followed by a Head comes up. What is the expectation of the number \( N_{TH} \) of flips we perform?

Solution.

\[ \text{Ex}[N_{TH}] = 4. \]

This time the tree diagram is \( C := H \cdot C + T \cdot B \) where the subtree \( B := H + T \cdot B \).

So

\[ \text{Ex}[N_{TH}] = (1 + \text{Ex}[N_{TH}]) \cdot \frac{1}{2} + (1 + \text{Ex}[N_B]) \cdot \frac{1}{2} \]

where \( N_B \) is the expected number of flips in the \( B \) subtree. But

\[ \text{Ex}[N_B] = 1 \cdot \frac{1}{2} + (1 + \text{Ex}[N_B]) \cdot \frac{1}{2}. \]

That is, \( \text{Ex}[N_B] = 2 \). Hence,

\[ \text{Ex}[N_{TH}] = \frac{1}{2} + \frac{\text{Ex}[N_{TH}]}{2} + \frac{1}{2} + \frac{2}{2} \]

which implies \( \text{Ex}[N_{TH}] = 4. \)
(c) Suppose we now play a game: flip a fair coin until either $TT$ or $TH$ first occurs. You win if $TT$ comes up first, lose if $TH$ comes up first. Since $TT$ takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you’re willing to play if you pay him $5 when he wins, but he merely pays you a 20% premium, that is, $6, when you win.

If you do this, you’re sneakily taking advantage of your opponent’s untrained intuition, since you’ve gotten him to agree to unfair odds. What is your expected profit per game?

**STAFF NOTE:** After the problem is solved, start a discussion of the apparent paradox: $TT$ and $TH$ are equally likely to show up first, but $TT$ takes longer to show up on average.

**Solution.** It’s easy to see that both $TT$ and $TH$ are equally likely to show up first. (Every game play consists of a sequence of $H$’s followed by a $T$, after which the game ends with a $T$ or an $H$, with equal probability.) So your expected profit is

$$\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot (-5)$$

dollars, that is 50 cents per game. So leap to play.

It may seem paradoxical $TT$ and $TH$ are equally likely to show up first, but $TT$ takes longer to show up on average. The explanation is that $TT$ takes longer to show up after $TH$ has appeared, than $TH$ takes after $TT$ has appeared. This is because when $TT$ appears, we’re already one step along the way to having $TH$ appear afterward, while when $TH$ shows up first, we start waiting for $TT$ to appear without a head start.

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**Problem 4.**

Let $T$ be a positive integer valued random variable such that

$$PDF_T(n) = \frac{1}{an^2},$$

where

$$a := \sum_{n \in \mathbb{Z}^+} \frac{1}{n^2}.$$ 

(a) Prove that $\text{Ex}[T]$ is infinite.

**Solution.**

$$\text{Ex}[T] := \sum_{n \in \mathbb{Z}^+} nPDF_T(n)$$

$$= \sum_{n \in \mathbb{Z}^+} n \frac{1}{an^2}$$

$$= \sum_{n \in \mathbb{Z}^+} \frac{1}{an}$$

$$= \frac{1}{a} \lim_{n \in \mathbb{Z}^+} H_n = \infty.$$ 

(b) Prove that $\text{Ex}[\sqrt{T}]$ is finite.
Solution.

\[
\text{Ex}[\sqrt{T}] = \sum_{n \in \mathbb{Z}^+} \sqrt{n} \cdot \frac{1}{an^2}
\]

\[
= \sum_{n \in \mathbb{Z}^+} \frac{1}{an^{3/2}}
\]

\[
< \frac{1}{a} \left( 1 + \int_{1}^{\infty} \frac{1}{n^{3/2}} \right)
\]

\[
= \frac{1}{a} + \frac{1}{a} \left( 0 - \frac{2}{\sqrt{a}} \right)
\]

\[
= \frac{3}{a} < \infty
\]

Note that if we define \( R = \sqrt{T} \), then \( R \) has finite expectation, but the variance of \( R \) is infinite.

Supplementary Problem

Problem 5.
Justify each line of the following proof that if \( R_1 \) and \( R_2 \) are independent, then

\[
\text{Ex}[R_1 \cdot R_2] = \text{Ex}[R_1] \cdot \text{Ex}[R_2].
\]

Proof.

\[
\text{Ex}[R_1 \cdot R_2] = \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r]
\]

\[
= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2]
\]

\[
= \sum_{r_1 \in \text{range}(R_1)} \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2]
\]

\[
= \sum_{r_1 \in \text{range}(R_1)} \left( r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right)
\]

\[
= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2]
\]

\[
= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1]
\]

\[
= \text{Ex}[R_2] \cdot \text{Ex}[R_1].
\]
Solution. Note that the event $[R_1 \cdot R_2 = r]$ is the disjoint union of events $[R_1 = r_1 \text{ AND } R_2 = r_2]$ such that $r_i \in \text{range}(R_i)$ for $i = 1, 2$ and $r_1r_2 = r$.

Proof.

\[
\begin{align*}
\text{Ex}[R_1 \cdot R_2] &:= \sum_{r \in \text{range}(R_1, R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \quad \text{(by definition)} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1r_2 \cdot \Pr[R_1 = r_1 \text{ AND } R_2 = r_2] \quad \text{(remarked above)} \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \quad \text{(ordering terms in the sum)} \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \left( \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \quad \text{(factor out } r_1 \Pr[R_1 = r_1]) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \quad \text{(def of Ex}[R_2]) \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \quad \text{(factor out Ex}[R_2]) \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1]. \quad \text{(def of Ex}[R_1])
\end{align*}
\]