Staff Solutions to In-Class Problems Week 11, Fri.

**STAFF NOTE:** Combinatorial Proof, Ch.14.10; Generating functions, Partial Fractions Ch.15.1–15.3  ■

**Problem 1.**
What do the following expressions equal? Give both algebraic and combinatorial proofs for your answers.

(a) \[ \sum_{i=0}^{n} \binom{n}{i} \]

**Solution.** \(2^n\).

*Algebraic proof:* This is the Binomial theorem with \(x = y = 1\).

*Combinatorial proof:* There are \(2^n\) length \(n\)-bit strings. The number of such sequences is also equal to the number of length \(n\)-bit strings with 0 ones, plus the number with 1 one, plus the number with 2 ones, etc., which is precisely \(\sum_{i=0}^{n} \binom{n}{i}\).  ■

(b) \[ \sum_{i=0}^{n} \binom{n}{i} (-1)^i \]

*Hint:* Consider the bit strings with an even number of ones and an odd number of ones.

**Solution.**

\[
\begin{cases} 
0 & \text{if } n > 0, \\
1 & \text{if } n = 0.
\end{cases}
\]

*Algebraic proof:* This is just the Binomial theorem with \(x = 1\) and \(y = -1\).

*Combinatorial proof:* Consider the \(n\)-bit strings, and divide them into two sets, those with an even number of ones (even \(i\) terms) and those with an odd number of ones (odd \(i\) terms). The sum is then equal to the number of strings with an even number of ones, minus the number of strings with an odd number of ones. Next, we note that for \(n > 0\), the number of \(n\)-bit strings with an even number of ones is equal to the number with an odd number of ones. This can be seen by establishing a bijection between the two sets: any string in one set can be made into a string in the other set by complementing the first bit in the string. Since the number of strings with an even number of ones is equal to the number with an odd number, the entire expression must be equal to 0.  ■
Problem 2.
We are interested in generating functions for the number of different ways to compose a bag of \( n \) donuts subject to various restrictions. For each of the restrictions in parts (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

Solution. There are no ways to select 0, 1, or 2 donuts, and one way to select \( n \) chocolate donuts for each \( n > 2 \), so the generating function is

\[
x^3 + x^4 + x^5 + \cdots = x^3 \left( 1 + x + x^2 + \cdots \right) = \frac{x^3}{1-x}
\]

(b) All the donuts are glazed and there are at most 2.

Solution. There is one way to select 0, 1, or 2 glazed donuts, and no ways to select \( n \) donuts for each \( n > 2 \), so the generating function is

\[
1 + x + x^2.
\]

(c) All the donuts are coconut and there are exactly 2 or there are none.

Solution.

\[
1 + x^2
\]

(d) All the donuts are plain and their number is a multiple of 4.

Solution. The generating function is

\[
1 + x^4 + x^8 + \cdots + x^{4n} + \cdots = \sum_{i=0}^{\infty} (x^4)^i = \frac{1}{1-x^4}
\]

(e) The donuts must be chocolate, glazed, coconut, or plain with the numbers of each flavor subject to the constraints above.

Solution. By the Convolution Rule, the generating function for selecting donuts with these constraints is the product of the preceding generating functions:

\[
\frac{x^3}{1-x} (1 + x + x^2)(1 + x^2) \cdot \frac{1}{1-x^4} = \frac{x^3(1 + x + x^2)(1 + x^2)}{(1-x)^2(1+x)(1+x^2)} = \frac{x^3(1 + x + x^2)}{(1-x)^2(1+x)}
\]
(f) Now find a closed form for the number of ways to select \( n \) donuts subject to the above constraints.

**Solution.** We would like to convert the generating function

\[
\frac{x^3(1 + x + x^2)}{(1 - x)^2(1 + x)}
\]

into partial fraction form. This requires that the numerator have lower degree than the denominator. We could accomplish this by expressing the ratio as a quotient and remainder, but in this case another simple approach applies. Namely, let

\[
G(x) := \frac{1 + x + x^2}{(1 - x)^2(1 + x)},
\]

so the generating function for donut selections is \( x^3G(x) \). Now we can express \( G(x) \) in partial fraction form and then use the fact that

\[
[x^n]x^3G(x) = [x^{n-3}]G(x)
\]

to obtain the generating function coefficients from the coefficients of \( G(x) \).

Expanding \( G(x) \) into partial fractions gives

\[
G(x) = \frac{A}{1 - x} + \frac{B}{(1 - x)^2} + \frac{C}{1 + x}
\]

for some constants, \( A, B, C \). We know that the coefficient of \( x^n \) in the series for \( (1 - x)^2 \) is, by the Convolution Rule, the number of ways to select \( n \) items of two different kinds, namely, \( \binom{n+1}{1} = n + 1 \), so we conclude that the \( n \)th coefficient in the series for \( G(x) \) is

\[
A + B(n + 1) + C(-1)^n. \tag{2}
\]

To find \( A, B, C \), we multiply both sides of (1) by the denominator \( (1 - x)^2(1 + x) \) to obtain

\[
1 + x + x^2 = A(1-x)(1+x) + B(1+x) + C(1-x)^2.
\]

Letting \( x = 1 \) in (3), we conclude that \( 3 = 2B \), so \( B = 3/2 \). Then, letting \( x = -1 \), we conclude \((-1)^2 = C2^2\), so \( C = 1/4 \). Finally, letting \( x = 0 \), we have

\[
1 = A + B + C = A + \frac{3}{2} + \frac{1}{4},
\]

so \( A = -3/4 \). Then from (2), we conclude that

\[
[x^n]G(x) = -\frac{3}{4} + \frac{3(n + 1)}{2} + \frac{(-1)^n}{4} = \frac{6n + 3 + (-1)^n}{4}.
\]

So the \( n \)th coefficient in the series for the generating function, \( x^3G(x) \), for donut selections is zero for \( n < 3 \), and, for \( n \geq 3 \), is \([x^{n-3}]G(x)\), namely,

\[
\frac{6(n - 3) + 3 + (-1)^{n-3}}{4} = \frac{6n - 15 - (-1)^n}{4}.
\]

**Supplemental problem**
Problem 3. (a) Let

\[ S(x) := \frac{x^2 + x}{(1 - x)^3}. \]

What is the coefficient of \( x^n \) in the generating function series for \( S(x) \)?

**Solution.** \( n^2 \). That is, \( S(x) = \sum_{n=1}^{\infty} n^2 x^n \).

To see why, note that the coefficient of \( x^n \) in \( 1/(1-x)^3 \) is, by the Convolution Rule, the number of ways to select \( n \) items of three different kinds, namely,

\[ \binom{n+2}{2} \cdot \frac{(n+2)(n+1)}{2}. \]

Now the coefficient of \( x^n \) in \( x^2/(1 - x)^3 \) is the same as the coefficient of \( x^{n-2} \) in \( 1/(1-x)^3 \), namely, \( ((n-2) + 2)((n-2) + 1)/2 = n(n-1)/2 \). Similarly, the coefficient of \( x^n \) in \( x/(1 - x)^3 \) is the same as the coefficient of \( x^{n-1} \) in \( 1/(1-x)^3 \), namely, \( ((n-1) + 2)((n-1) + 1)/2 = (n+1)n/2 \). The coefficient of \( x^n \) in \( S(x) \) is the sum of these two coefficients, namely,

\[ \frac{n(n-1)}{2} + \frac{(n+1)n}{2} = \frac{n^2 - n + n^2 + n}{2} = n^2. \]

(b) Explain why \( S(x)/(1 - x) \) is the generating function for the sums of squares. That is, the coefficient of \( x^n \) in the series for \( S(x)/(1 - x) \) is \( \sum_{k=1}^{n} k^2 \).

**Solution.**

\[
\left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k \cdot 1 \right) x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k \right) x^n
\]

by the convolution formula for the product of series. For \( S(x) \), the coefficient of \( x^k \) is \( a_k = k^2 \), and

\[ S(x)/(1 - x) = S(x) \left( \sum_{n=0}^{\infty} x^n \right), \]

so (4) implies that the coefficient of \( x^n \) in \( S(x)/(1 - x) \) is the sum of the first \( n \) squares.

(c) Use the previous parts to prove that

\[ \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

**Solution.** We have

\[
\frac{S(x)}{1 - x} = \frac{\frac{x(1+x)}{(1-x)^2}}{1 - x} = \frac{x + x^2}{(1 - x)^4}
\]

The coefficient of \( x^n \) in the series expansion of \( 1/(1-x)^4 \) is

\[ \binom{n+2}{2} \cdot \frac{(n+2)(n+1)}{2}. \]
But by (5),

\[
\frac{S(x)}{1 - x} = \frac{x}{(1 - x)^4} + \frac{x^2}{(1 - x)^4},
\]

so the coefficient of \(x^n\) is the sum of the \((n - 1)\)st and \((n - 2)\)nd coefficients of \((1 - x)^4\), namely,

\[
\frac{n(n + 1)(n + 2)}{3!} + \frac{(n - 1)n(n + 1)}{3!} = \frac{n(n + 1)(2n + 1)}{6}.
\]

**Problem 4.**

T-Pain is planning an epic boat trip and he needs to decide what to bring with him.

- He must bring some burgers, but they only come in packs of 6.
- He and his two friends can’t decide whether they want to dress formally or casually. He’ll either bring 0 pairs of flip flops or 3 pairs.
- He doesn’t have very much room in his suitcase for towels, so he can bring at most 2.
- In order for the boat trip to be truly epic, he has to bring at least 1 nautical-themed pashmina afghan.

(a) Let \(B(x)\) be the generating function for the number of ways to bring \(n\) burgers, \(F(x)\) for the number of ways to bring \(n\) pairs of flip flops, \(T(x)\) for towels, and \(A(x)\) for Afghans. Write simple formulas for each of these.

\[
B(x) = \quad F(x) = \\
T(x) = \quad A(x) = 
\]

**Solution.**

\[
B(x) = \frac{x^6}{1 - x^6}, \\
F(x) = 1 + x^3, \\
T(x) = 1 + x + x^2 = \frac{1 - x^3}{1 - x} \\
A(x) = \frac{x}{1 - x}. 
\]

(b) Let \(g_n\) be the number of different ways for T-Pain to bring \(n\) items (burgers, pairs of flip flops, towels, and/or afghans) on his boat trip. Let \(G(x)\) be the generating function \(\sum_{n=0}^{\infty} g_n x^n\). Verify that

\[
G(x) = \frac{x^7}{(1 - x)^2}.
\]
Solution. By the Convolution Rule 15.2.3,

\[
G(x) = B(x) F(x) T(x) A(x) \\
= \frac{x^6}{1-x^6} \cdot \frac{1-x^3}{1-x} \cdot \frac{x}{1-x} \\
= \frac{x^6(1+x^3)(1-x^3)x}{(1-x^6)(1-x)^2} \\
= \frac{x^7}{(1-x)^2}
\]

(e) Find a simple formula for \( g_n \).

Solution.

\[
g_n = \begin{cases} 
0 & \text{for } n < 7 \\
n - 6 & \text{for } n \geq 7.
\end{cases}
\]

Let

\[
H(x) := \frac{1}{(1-x)^2},
\]

so \( G(x) = x^7 H(x) \). We know that the coefficient, \( h_n \), of \( x^n \) in the series for \( H(x) \) is, by the Convolution Rule, the number of ways to select \( n \) items of two different kinds, namely, \( h_n = \binom{n+1}{1} = n + 1 \). So we conclude that for \( n \geq 7 \), the \( n \)th coefficient in the series for \( G(x) \) is \( h_{n-7} \) namely (6).