Staff Solutions to In-Class Problems Week 10, Fri.

**STAFF NOTE:** Generalized Rules for Counting, Binomial Theorem, Bookkeeper Rule, Multinomial Theorem, Ch. 14.3-14.7

**Problem 1.**
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a $k$-to-1 mapping for what $k$?

**Solution.** Two lists map to the same sequence of groups iff the first 3 students are the same on both lists, and likewise for the second, third, and fourth consecutive sublists of 3 students. So for a given sequence of 4 groups, the number of lists which map to it is 

\[(3!)^4\]

because there are 3! ways to order the students in each of the 4 consecutive sublists.

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

\[\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\}\]

into a group assignment

\[\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\],

this mapping is $j$-to-1 for what $j$?

**Solution.** $4!$.
Each of the $4!$ sequences of a particular set of four groups maps to that set of groups.

(c) How many group assignments are possible?

**Solution.**

\[
\frac{12!}{4! \cdot (3!)^4} = 15400
\]

different assignments.

There are 12! possible lists of students, and we can map each list to an assignment by first mapping the list to a sequence of four groups, and then mapping the sequence to the assignment. Since the first map is $(3!)^4$-to-1 and the second is $4!$-to-1, the composite map is $(3!)^4 \cdot 4!$-to-1. So by the Division Rule, $12! = \left((3!)^4 \cdot 4!\right) A$ where $A$ is the number of assignments.
(d) In how many ways can $3n$ students be broken up into $n$ groups of 3?

**Solution.**

\[
\frac{(3n)!}{(3!)^n n!}
\]

This follows simply by replacing “12” by “3n” and “4” by “n” in the solution to the previous problem parts.

**Problem 2.** (a) There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

**Solution.** The answer is the number of length 16 bitstrings with eight 1’s, namely

\[
\binom{16}{8}
\]

As in Problem 14.6, there is an obvious bijection between length 30 bit-strings with eight 1’s and selections of eight among 30 books on a shelf. And there is another obvious bijection between length 30 bit-strings with eight 1’s that are at least two apart and length 16 bit-strings with eight 1’s, namely, insert two 0’s after the first seven 1’s in a 16 bit-string with eight 1’s, to obtain a 30 bit-string with 8 occurrences of 1’s that are at least two apart.

(b) How many nonnegative integer solutions are there for the following equality?

\[
x_1 + x_2 + \cdots + x_m = k.
\]

**Solution.** There are

\[
\binom{m + k - 1}{k}
\]

nonnegative integer solutions to (1).

As in Problem 14.7, mapping \((x_1, x_2, \ldots, x_m) \in \mathbb{N}^m\) to \(0^{x_1}10^{x_2}1\ldots0^{x_m}\) defines a bijection between solutions to (1) and length \(k + m - 1\) bit strings with \(k\) 0’s.

(c) How many nonnegative integer solutions are there for the following inequality?

\[
x_1 + x_2 + \cdots + x_m \leq k.
\]

**Solution.** There are

\[
\binom{m + k}{k}
\]

nonnegative integer solutions to (2).

There is an obvious bijection between integer solutions to the equality

\[
x_1 + x_2 + \cdots + x_m + x_{m+1} = k,
\]

and solutions to (2), and the answer follows as in part (b).

(d) How many length $m$ weakly increasing sequences of nonnegative integers \(\leq k\) are there?
Solution. There are
\[
\binom{m + k}{k}
\]
length \(m\) weakly increasing sequences of nonnegative integers \(\leq k\).
As in Problem 14.7, there is a bijection between these weakly increasing sequences and solutions to (2), namely, map a solution \((x_1, x_2, \ldots, x_m)\) to a sequence \((x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, \sum_{i=1}^{m} x_i)\). So the answer follows from part (b).

Problem 3.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word \(BOOKKEEPER\).
(a) In how many ways can you arrange the letters in the word \(POKE\)?

Solution. There are \(4!\) arrangements corresponding to the \(4!\) permutations of the set \(\{P, O, K, E\}\).

(b) In how many ways can you arrange the letters in the word \(BO_1 O_2 K\)? Observe that we have subscripted the O’s to make them distinct symbols.

Solution. There are \(4!\) arrangements corresponding to the \(4!\) permutations of the set \(\{B, O_1, O_2, K\}\).

(c) Suppose we map arrangements of the letters in \(BO_1 O_2 K\) to arrangements of the letters in \(BOOK\) by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

\[
\begin{align*}
O_2 BO_1 K & \quad \rightarrow & \quad BOOK \\
KO_2 BO_1 & \quad \rightarrow & \quad OBOK \\
O_1 BO_2 K & \quad \rightarrow & \quad KOBO \\
KO_1 BO_2 K & \quad \rightarrow & \quad \ldots \\
BO_1 O_2 K & \quad \rightarrow & \quad \ldots \\
BO_2 O_1 K & \quad \rightarrow & \quad \ldots \\
\ldots & \quad \rightarrow & \quad \ldots
\end{align*}
\]

(d) What kind of mapping is this, young grasshopper?

Solution. 2-to-1

(e) In light of the Division Rule, how many arrangements are there of \(BOOK\)?

Solution. \(4!/2\)

(f) Very good, young master! How many arrangements are there of the letters in \(KE_1 E_2 PE_3 R\)?

Solution. \(6!\)

(g) Suppose we map each arrangement of \(KE_1 E_2 PE_3 R\) to an arrangement of \(KEEPER\) by erasing subscripts. List all the different arrangements of \(KE_1 E_2 PE_3 R\) that are mapped to \(REPEEK\) in this way.
Solution. $RE_1PE_2E_3K, RE_1PE_3E_2K, RE_2PE_1E_3K, RE_2PE_3E_1K, RE_3PE_1E_2K, RE_3PE_2E_1K$

(h) What kind of mapping is this?

Solution. $3!$-to-1

(i) So how many arrangements are there of the letters in $KEEPER$?

Solution. $6!/3!$

Now you are ready to face the BOOKKEEPER!

(j) How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

Solution. $10!$

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

Solution. $10!/2!$

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

Solution. $10!/(2! \cdot 2!)$

(m) How many arrangements of $BOOKKEEPER$ are there?

Solution.

$$
\binom{10}{1, 2, 2, 3, 1, 1} := \frac{10!}{1! \cdot 2! \cdot 3! \cdot 1! \cdot 1!} = \frac{10!}{(2!)^2 \cdot 3!}
$$

Remember well what you have learned: subscripts on, subscripts off. This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

Solution.

$$
\binom{10}{1, 2, 5, 2} := \frac{10!}{1! \cdot 2! \cdot 5! \cdot 2!}
$$

(o) How many length 52 sequences of digits contain exactly 17 two’s, 23 fives, and 12 nines?

Solution.

$$
\binom{52}{17, 23, 12} := \frac{52!}{17! \cdot 23! \cdot 12!}
$$
Problem 4.
Solve the following counting problems. Define an appropriate mapping (bijective or \(k\)-to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

Solution. There is a bijection from sequences containing one \(P\), two \(K\)’s, three \(B\)’s, a \(C\), and two \(D\)’s. In any such sequence, the letter in the \(i\)th position specifies the task assigned to the \(i\)th candidate. Therefore, the number of possible assignments is:

\[
\binom{9}{1, 2, 3, 1, 2} := \frac{9!}{1! 2! 3! 1! 2!}
\]

(b) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

Solution. We identify the nonnegative integers less than 1,000,000 with the length 6 strings of decimal digits. Then there is a bijection with pairs:

(position of the 9, successive values of other 5 digits)

The sum of the other 5 digits is equal to 8, so the number of ways to choose their values is equal to the number of solutions over the nonnegative integers to

\[x_1 + x_2 + x_3 + x_4 + x_5 = 8,\]  

namely, \(\binom{12}{4}\). So by the product rule there are

\[6 \cdot \binom{12}{4}\]

such integers.

Supplemental problem

Problem 5. (a) Use the Multinomial Theorem 14.6.5 to prove that

\[(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}\]  

for all primes \(p\). (Do not prove it using Fermat’s “little” Theorem. The point of this problem is to offer an independent proof of Fermat’s theorem.)

Hint: Explain why \(\binom{k_1, k_2, \ldots, k_n}{p}\) is divisible by \(p\) if all the \(k_i\)’s are positive integers less than \(p\).
**Solution.** By the Multinomial Theorem 14.6.5, \((x_1 + x_2 + \cdots + x_n)^p\) is a sum of monomials in \(x_1, \ldots, x_n\) whose coefficients are 
\[
\binom{p}{k_1, k_2, \ldots, k_n}
\]
where the sum of the \(k_i\)'s is \(p\). But if all the \(k_i\)'s are less than \(p\), then none of the denominator terms divides the numerator, \(p\), and so the multinomial coefficient is divisible by \(p\). So the only coefficients not divisible by \(p\) are the coefficients of the terms \(x_i^p\), and all the other terms are \(\equiv 0 \pmod{p}\).

**(b)** Explain how (4) immediately proves Fermat’s Little Theorem 8.10.8: \(n^{p-1} \equiv 1 \pmod{p}\) when \(n\) is not a multiple of \(p\).

**Solution.** Let \(x_1 = x_2 = \cdots = x_n = 1\). Then (4) implies \(n^p \equiv n \cdot 1^p = n \pmod{p}\). If \(n\) is not a multiple of \(p\), then we can then cancel \(n\) to get \(n^{p-1} \equiv 1 \pmod{p}\).