Notes for Recitation 8

Routing in a Beneš Network

In lecture, we saw that the Beneš network has a max congestion of 1; that is, every permutation can be routed in such a way that a single packet passes through each switch. Let’s work through an example. A Beneš network of size $N = 8$ is attached.

1. Within the Beneš network of size $N = 8$, there are two subnetworks of size $N = 4$. Put boxes around these. Hereafter, we’ll refer to these as the upper and lower subnetworks.

Solution.

2. Now consider the following permutation routing problem:

$$
\begin{align*}
\pi(0) &= 3 & \pi(4) &= 2 \\
\pi(1) &= 1 & \pi(5) &= 0 \\
\pi(2) &= 6 & \pi(6) &= 7 \\
\pi(3) &= 5 & \pi(7) &= 4
\end{align*}
$$

Each packet must be routed through either the upper subnetwork or the lower subnetwork. Construct a graph with vertices 0, 1, \ldots, 7 and draw a dashed edge between each pair of packets that can not go through the same subnetwork because a collision would occur in the second column of switches.
3. Add a solid edge in your graph between each pair of packets that cannot go through the same subnetwork because a collision would occur in the next-to-last column of switches.

Solution.

4. Color (i.e., label) the vertices of your graph red and blue so that adjacent vertices get different colors. Why must this be possible, regardless of the permutation $\pi$?
Solution. This must be possible, because the dashed edges form a matching and the solid edges form another matching. Based on the proof in Appendix 1, we know that these two matchings will form a bipartite graph. As seen in Recitation 5, a bipartite graph is 2-colorable. Hence, we can color the vertices red and blue, regardless of the permutation $\pi$.

5. Suppose that red vertices correspond to packets routed through the upper subnetwork and blue vertices correspond to packets routed through the lower subnetwork. On the attached copy of the Beneš network, highlight the first and last edge traversed by each packet.

Solution.

6. All that remains is to route packets through the upper and lower subnetworks. One way to do this is by applying the procedure described above recursively on each subnetwork. However, since the remaining problems are small, see if you can complete all the paths on your own.

Solution.
7. Prove that the congestion of the $N$-input Beneš network is 1 for any $N$ that is a power of 2.

Solution. See the proof for Theorem 6.3.2 in the textbook.
Appendix 1
Let $G = (V, E)$ be a graph. A matching in $G$ is a set $M \subseteq E$ such that no two edges in $M$ are incident on a common vertex.

Let $M_1, M_2$ be two matchings of $G$. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either $M_1$ or $M_2$). We show that $G'$ is bipartite.

Helpful definition: A connected component is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

Proof. Proof by induction on the number of vertices $n$:

Induction hypothesis: $P(n)$ is defined to be: Let $G$ be a graph with $n$ vertices and matchings $M_1$ and $M_2$. Let $G' = (V, M_1 \cup M_2)$. Then $G'$ is bipartite.

Base case: $G$ has only one vertex and so is bipartite. $P(1)$ holds.

Inductive step: We will assume $P(n)$ in order to prove $P(n + 1)$.

Let $G$ be a graph with $n + 1$ vertices. We will remove a vertex $v$ from $G$ to obtain an $n$ vertex graph, $G_1$, with vertex set $V_1$. If we remove $v$ we will be in one of the following cases:

Case 1: $v$ is in none of the edges in $M_1$ nor $M_2$.

By our inductive hypothesis we know that since $G_1$ has $n$ vertices and $M_1$ and $M_2$ are matchings of $G_1$, then $G'_1 = (V_1, M'_1 \cup M_2)$ is bipartite. Since $G'_1$ is bipartite, there exists a partition of the vertices into two sets, $L$ and $R$ such that every edge is incident to a vertex in $L$ and to a vertex in $R$. We can now add $v$ to either set and obtain a bipartite representation of $G'$.

Case 2: $v$ is in an edge in either $M_1$ or $M_2$, we will assume, without loss of generality, that $v$ is in an edge in $M_1$.

Suppose the edge $v - x$ is in $M_1$, now remove $v - x$ from $M_1$ to obtain $M'_1$.

Now by our inductive hypothesis we know that since $G_1$ has $n$ vertices and $M'_1$ and $M_2$ are matchings of $G_1$, then $G'_1 = (V_1, M'_1 \cup M_2)$ is bipartite. Since $G'_1$ is bipartite, there exists a partition of the vertices into two sets, $L$ and $R$ such that every edge is incident to a vertex in $L$ and to a vertex in $R$.

We know that the vertex $x$ is in either $L$ or $R$. We can just add vertex $v$ to the other set, along with edge $v - x$, and we obtain a valid partitioning of $L$ and $R$ for our graph $G'$.

Case 3: $v$ is in both $M_1$ and $M_2$

Suppose the edge $v - x$ is in $M_1$ and $v - y$ is in $M_2$, now remove those edges from $M_1$ and $M_2$ to obtain $M'_1$ and $M'_2$.

Now by our inductive hypothesis we know that since $G_1$ has $n$ vertices and $M'_1$ and $M'_2$ are matchings of $G_1$, then $G'_1 = (V_1, M'_1 \cup M'_2)$ is bipartite. Since $G'_1$ is bipartite, there exists a partition of the vertices into two sets, $L$ and $R$ such that every edge is incident to a vertex in $L$ and to a vertex in $R$. 
If $x$ and $y$ in the same set, either $L$ or $R$, then we can just add $v$ to the other set, and add edges $v - x$ and $v - y$ to obtain $G'$. So our graph remains bipartite.

If $x$ and $y$ are on different sides of $L$ and $R$, then either $x$ and $y$ are in the same connected component or they are in different connected components. If $x$ and $y$ are in different connected components, then each connected component has a corresponding set of $L$ and $R$ vertices, such that there are edges only within that component. Let’s say that the first component has left and right vertices in the set $L_1$ and $R_1$, and the second component has sets $L_2$ and $R_2$, where $L = L_1 \cup L_2$ and $R = R_1 \cup R_2$. Now if we swap $L_1$ and $R_1$ – that is we define $L = R_1 \cup L_2$ and $R = L_1 \cup R_2$ – then our graph will remain bipartite, as there were edges only within the connected components. But after the swapping $x$ and $y$ will be in the same set, $L$ or $R$, and as before we can just add $v$ to the other set to get a bipartite graph for $G'$ as desired.

Now we will show that it is impossible for $x$ and $y$ to be on opposite sides and in the same component.

Suppose for a contradiction that $x$ and $y$ are in the same connected component and $x$ and $y$ are both in $L$. Then since they are in the same connected component there is a path from $x$ to $y$ say $x - v_1, v_1 - v_2, \ldots, v_k - y$, where $k$ is even. Then the edges $x - v_1$ and $v_2 - v_3, v_4 - v_5, \ldots, v_k - y$ must all be in the same matching (otherwise we will have two edges incident on the same vertex in the same matching). This is a contradiction since our original $M_1$ cannot have any edge with $x$ and $M_2$ cannot have any edge with $y$ (since a matching has only one edge incident to a vertex). So this cannot be the case and $x$ and $y$ must be on the same side.

Hence we conclude that in all cases $G'$ is bipartite.

Therefore by induction our claim holds. \[\Box\]