Notes for Recitation 25

1 Expected Value Rule for Functions of Random Variables

In lecture, we have computed the expectation of a function of a random variable without explicitly discussing the general rule. For example, yesterday we saw that the expectation of the square of the roll of a die is not equal to the square of the expectation of the roll. That is, if $R$ is the outcome of a single roll of a die. Then $\text{Ex}(R^2) \neq \text{Ex}(R)^2$. We will now explicitly present the rule for computing the expectation of a function of a random variable $R$.

**Rule** (Expected Value for the Function of a Random Variable). Let $R$ be a random variable, and let $f(R)$ be a function of $R$. Then, the expected value of the random variable $f(R)$ is given by

$$
E[f(R)] = \sum_{x \in \text{Range}(R)} f(x) \cdot \Pr\{R = x\}
$$

2 Properties of Variance

In this problem we will study some properties of the variance and the standard deviation of random variables.


**Solution.** Let $\mu = E[R]$. Then

$$
\text{Var}[R] = E[(R - E[R])^2] \\
= E[(R - \mu)^2] \quad \text{(Definition of variance)} \\
= E[R^2 - 2\mu R + \mu^2] \quad \text{(def. of } \mu) \\
= E[R^2] - 2\mu E[R] + \mu^2 \quad \text{(linearity of expectation)} \\
= E[R^2] - 2\mu^2 + \mu^2 \quad \text{(def. of } \mu) \\
= E[R^2] - \mu^2 \quad \text{(def. of } \mu) \\
= E[R^2] - E^2[R]. \quad \text{(def. of } \mu)
$$
b. Show that for any random variable \( R \) and constants \( a \) and \( b \), \( \text{Var} [aR + b] = a^2 \text{Var} [R] \). Conclude that the standard deviation of \( aR + b \) is \( a \) times the standard deviation of \( R \).

**Solution.** We will transform the left side into the right side. The first step is to expand \( \text{Var} [aR + b] \) using the alternate definition of variance.

\[
\text{Var} [aR + b] = \text{E} [(aR + b)^2] - \text{E}^2 [aR + b].
\]

We will work on the first term and then the second term. For the first term, note that by linearity of expectation,

\[
\text{E} [(aR + b)^2] = \text{E} [a^2 R^2 + 2abR + b^2] = a^2 \text{E} [R^2] + 2ab \text{E} [R] + b^2.
\]

Similarly for the second term:

\[
\text{E}^2 [aR + b] = (a \text{E} [R] + b)^2 = a^2 \text{E}^2 [R] + 2ab \text{E} [R] + b^2.
\]

Finally, we subtract the expanded second term from the first.

\[
\text{Var} [aR + b] = \text{E} [(aR + b)^2] - \text{E}^2 [aR + b] = a^2 \text{E} [R^2] + 2ab \text{E} [R] + b^2 - (a^2 \text{E}^2 [R] + 2ab \text{E} [R] + b^2) = a^2 (\text{E} [R^2] - \text{E}^2 [R]) = a^2 \text{Var} [R].
\]

Since the standard deviation of a random variable is the square root of the variance, the standard deviation of \( aR + b \) is \( \sqrt{a^2 \text{Var} [R]} \) which is just \( a \) times the standard deviation of \( R \). 

\[\blacksquare\]

c. Show that if \( R_1 \) and \( R_2 \) are independent random variables, then

\[
\text{Var} [R_1 + R_2] = \text{Var} [R_1] + \text{Var} [R_2].
\]

**Solution.** We will transform the left side into the right side. We begin by applying the alternate definition of variance.

\[
\text{Var} [R_1 + R_2] = \text{E} [(R_1 + R_2)^2] - \text{E}^2 [R_1 + R_2].
\]

We will work on the first term and then the second term separately. For the first term, note

\[
\text{E} [(R_1 + R_2)^2] = \text{E} [R_1^2 + 2R_1 R_2 + R_2^2] = \text{E} [R_1^2] + \text{E} [2R_1 R_2] + \text{E} [R_2^2] = \text{E} [R_1^2] + 2 \text{E} [R_1] \text{E} [R_2] + \text{E} [R_2^2].
\]
First, we multiply out the squared expression. The second step uses linearity of expectation. In the last step, we break the expectation of the product $R_1 R_2$ into a product of expectations; this is where we use the fact that $R_1$ and $R_2$ are independent.

Now we work on the second term.

\[
E^2 [R_1 + R_2] = (E [R_1] + E [R_2])^2
= E^2 [R_1] + 2 E [R_1] E [R_2] + E^2 [R_2].
\]

The first step uses linearity of expectation, and in the second step we multiply out the squared expression. Now we subtract the (expanded) second term from the first. Cancelling and rearranging terms, we find that

\[
\text{Var} [R_1 + R_2] = (E [R_2] - E^2 [R_1]) + (E [R_2^2]) - E^2 [R_2])
= \text{Var} [R_1] + \text{Var} [R_2].
\]

\[\square\]

d. Give an example of random variables $R_1$ and $R_2$ for which $\text{Var} [R_1 + R_2] \neq \text{Var} [R_1] + \text{Var} [R_2]$.

**Solution.** Suppose $R = R_1 = R_2$. If linearity of variance held, then $\text{Var} [R + R] = \text{Var} [R] + \text{Var} [R]$. However, by part b, $\text{Var} [R + R] = \text{Var} [2R] = 4 \text{Var} [R]$. This is only possible if $\text{Var} [R] = 0$. If, say, we choose $R$ to be the outcome of a fair coin flip, $\text{Var} [R] \neq 0$. In fact, any $R$ which holds at least 2 distinct values each with positive probability will do. \[\square\]

e. Compute the variance and standard deviation of the Binomial distribution $H_{n,p}$ with parameters $n$ and $p$.

**Solution.** We know that $H_{n,p} = \sum_{k=1}^{n} I_k$ where the $I_k$ are mutually independent 0-1-valued variables with $\Pr \{I_k = 1\} = p$. The variance of $I_k$ is $E [I_k^2] - E [I_k]^2 = E [I_k] - E [I_k]^2 = E [I_k] (1 - E [I_k]) = p(1 - p)$. Thus, by linearity of variance, we have $\text{Var} [H_{n,p}] = n \text{Var} [I_k] = np(1 - p)$. Thus, the standard deviation of $H_{n,p}$ is $\sqrt{np(1 - p)}$. \[\square\]

f. Let’s say we have a random variable $T$ such that $T = \sum_{j=1}^{n} T_j$, where all of the $T_j$’s are mutually independent and take values in the range $[0, 1]$. Prove that $\text{Var}(T) \leq E(T)$. We’ll use this result in lecture tomorrow. **Hint:** Upper bound $\text{Var} [T_j]$ with $E [T_j]$ using the definition of variance in part (a) and the rule for computing the expectation of a function of a random variable.

**Solution.** We know by linearity of variance for mutually-independent random variables that
\[
\text{Var}[T] = \text{Var}[T_1 + \ldots + T_n] \\
= \text{Var}[T_1] + \ldots + \text{Var}[T_n]
\]

Now we evaluate the variance of \( T_j \) for any \( j \). Using the definition of variance from part (a) above, we have

\[
\text{Var}[T_j] = E[T_j^2] - E[T_j]^2
\]

By the rule for computing the expectation of a function of a random variable, we also know that

\[
E[T_j^2] = \sum_{x \in \text{Range}(T_j)} x^2 \Pr\{T_j = x\}
\]

Now we can use the fact that \( T_j \) is in the range \([0, 1]\) to say that \( x^2 \leq x \), and that therefore

\[
\sum_{x \in \text{Range}(T_j)} x^2 \cdot \Pr\{T_j = x\} \leq \sum_{x \in \text{Range}(T_j)} x \cdot \Pr\{T_j = x\}
\]

Thus, \( E[T_j^2] \leq E[T_j] \) and

\[
\text{Var}[T_j] \leq E[T_j] - E[T_j]^2 \\
\leq E[T_j]
\]

Since this holds for any \( j \), we can now conclude that

\[
\text{Var}[T] = \text{Var}[T_1] + \ldots + \text{Var}[T_n] \\
\leq E[T_1] + \ldots + E[T_n] \\
= E[T]
\]

\[\blacksquare\]