Notes for Recitation 18

The \textit{(ordinary) generating function} for a sequence $\langle a_0, a_1, a_2, a_3, \ldots \rangle$ is the power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

\textbf{Solution.}

$$2 + 3x + 5x^2$$

\[\square\]

(b) $\langle 1, 1, 1, 1, 1, 1, \ldots \rangle$

\textbf{Solution.}

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

\[\square\]

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

\textbf{Solution.}

$$1 + 2x + 4x^2 + 8x^3 + \ldots = (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots$$

$$= \frac{1}{1-2x}$$

\[\square\]

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

\textbf{Solution.}

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$

\[\square\]

(e) $\langle 0, 0, 0, 1, 1, 1, 1, \ldots \rangle$
Solution.

\[ x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) = \frac{x^3}{1 - x} \]

(f) \( \langle 1, 3, 5, 7, 9, 11, \ldots \rangle \)

Solution.

\[
\begin{align*}
1 + x + x^2 + x^3 + \ldots &= \frac{1}{1 - x} \\
\frac{d}{dx} (1 + x + x^2 + x^3 + \ldots) &= \frac{d}{dx} \frac{1}{1 - x} \\
1 + 2x + 3x^2 + 4x^3 + \ldots &= \frac{1}{(1 - x)^2} \\
2 + 4x + 6x^2 + 8x^3 + \ldots &= \frac{2}{(1 - x)^2} \\
1 + 3x + 5x^2 + 7x^3 + \ldots &= \frac{2}{(1 - x)^2} = \frac{1}{1 - x} \\
&= \frac{1 + x}{(1 - x)^2}
\end{align*}
\]
Problem 2

Suppose that:

\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \]
\[ g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \]

What sequences do the following functions generate?

(a) \( f(x) + g(x) \)

Solution.

\[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \cdots\]

(b) \( f(x) \cdot g(x) \)

Solution.

\[ a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots + \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n \]

(c) \( f(x)/(1-x) \)

Solution. This is a special case of the preceding problem part where:

\[ g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \]

and so \( b_0 = b_1 = b_2 = \ldots = 1 \). In this case, we have:

\[ f(x) \cdot g(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + \left( \sum_{k=0}^{n} a_k \right) x^n \]

Thus, \( f(x)/(1-x) \) is the generating function for sums of prefixes of the sequence generated by \( f \).
Problem 3

There is a jar containing \( n \) different flavors of candy. I’d like to pick out a set of \( k \) candies.

(a) In how many different ways can this be done?

**Solution.** There is a bijection with sequences containing \( k \) zeroes (representing candies) and \( n - 1 \) ones (separating the different varieties). The number of such sequences is:

\[
\binom{n+k-1}{k}
\]

(b) Now let’s approach the same problem using generating functions. Give a closed-form generating function for the sequence \( \langle s_0, s_1, s_2, s_3, \ldots \rangle \) where \( s_k \) is the number of ways to select \( k \) candies when there is only \( n = 1 \) flavor available.

**Solution.**

\[
1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}
\]

(c) Give a closed-form generating function for the sequence \( \langle t_0, t_1, t_2, t_3, \ldots \rangle \) where \( t_k \) is the number of ways to select \( k \) candies when there are \( n = 2 \) flavors.

**Solution.**

\[
(1 + x + x^2 + x^3 + \ldots)^2 = \frac{1}{(1-x)^2}
\]

(d) Give a closed-form generating function for the sequence \( \langle u_0, u_1, u_2, u_3, \ldots \rangle \) where \( u_k \) is the number of ways to select \( k \) candies when there are \( n \) flavors.

**Solution.**

\[
\frac{1}{(1-x)^n}
\]
Problem 4

Consider the following recurrence equation:

\[
T_n = \begin{cases} 
1 & n = 0 \\
2 & n = 1 \\
2T_{n-1} + 3T_{n-2} & (n \geq 2)
\end{cases}
\]

Let \( f(x) \) be a generating function for the sequence \( \langle T_0, T_1, T_2, T_3, \ldots \rangle \).

(a) Give a generating function in terms of \( f(x) \) for the sequence:

\[\langle 1, 2, 2T_1 + 3T_0, 2T_2 + 3T_1, 2T_3 + 3T_2, \ldots \rangle\]

**Solution.** We can break this down into a linear combination of three sequences:

\[
\begin{align*}
\langle 1, 2, 0, 0, 0, \ldots \rangle &= 1 + 2x \\
\langle 0, T_0, T_1, T_2, T_3, \ldots \rangle &= xf(x) \\
\langle 0, 0, T_0, T_1, T_2, \ldots \rangle &= x^2f(x)
\end{align*}
\]

In particular, the sequence we want is very nearly generated by \( 1 + 2x + 2xf(x) + 3x^2f(x) \). However, the second term is not quite correct; we’re generating \( 2 + 2T_0 = 4 \) instead of the correct value, which is 2. We correct this by subtracting \( 2x \) from the generating function, which leaves:

\[1 + 2xf(x) + 3x^2f(x)\]

(b) Form an equation in \( f(x) \) and solve to obtain a closed-form generating function for \( f(x) \).

**Solution.** The equation

\[f(x) = 1 + 2xf(x) + 3x^2f(x)\]

equates the left sides of all the equations defining the sequence \( T_0, T_1, T_2, \ldots \) with all the right sides. Solving for \( f(x) \) gives the closed-form generating function:

\[f(x) = \frac{1}{1 - 2x - 3x^2}\]

(c) Expand the closed form for \( f(x) \) using partial fractions.
Solution. We can write:

\[ 1 - 2x - 3x^2 = (1 + x)(1 - 3x) \]

Thus, there exist constants \(A\) and \(B\) such that:

\[
f(x) = \frac{1}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x}
\]

Now substituting \(x = 0\) and \(x = 1\) gives the system of equations:

\[
\begin{align*}
1 &= A + B \\
-\frac{1}{4} &= \frac{A}{2} - \frac{B}{2}
\end{align*}
\]

Solving the system, we find that \(A = \frac{1}{4}\) and \(B = \frac{3}{4}\). Therefore, we have:

\[
f(x) = \frac{1/4}{1 + x} + \frac{3/4}{1 - 3x}
\]

(d) Find a closed-form expression for \(T_n\) from the partial fractions expansion.

Solution. Using the formula for the sum of an infinite geometric series gives:

\[
f(x) = \frac{1}{4} \left(1 - x + x^2 - x^3 + x^4 - \ldots\right) + \frac{3}{4} \left(1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + \ldots\right)
\]

Thus, the coefficient of \(x^n\) is:

\[
T_n = \frac{1}{4} \cdot (-1)^n + \frac{3}{4} \cdot 3^n
\]