Practice Quiz 2 Solutions

Problem 1. [10 points] Consider the following relation on the set of natural numbers:

\[ R = \{ (x, y) : x \leq y^2 \text{ for } x, y \in \mathbb{N} \}. \]

Which of the following properties holds for \( R \)? If it has the property, prove it. If not, provide a counterexample.

1. reflexive.

Solution. Yes. \( R \) is reflexive if \( \forall x \ xRx \), that is, if \( x \leq x^2 \). If \( x = 0 \), \( 0 \leq 0 \), so reflexivity holds. If \( x \geq 1 \), then it follows that \( x^2 \geq x \), from multiplying both sides of the inequality by \( x \), which is positive. This covers all \( x \in \mathbb{N} \). □

2. symmetric.

Solution. No. Counterexample: \( x = 0, y = 1 \). \( 0 \leq 1^2 \), but \( 1 > 0^2 \). □

3. transitive.

Solution. No. Counterexample: \( x = 10, y = 5, z = 3 \). \( 10 \leq 5^2 \), and \( 5 \leq 3^2 \), but \( 10 \geq 3^2 \). □

4. antisymmetric.

Solution. No. Counterexample: \( x = 2, y = 3 \). \( 2 \leq 3^2 \), and \( 3 \leq 2^2 \), but \( 3 \neq 2 \). □

5. equivalence relation.

Solution. No. An equivalence relation must be reflexive, symmetric, and transitive. Of those three, the relation is only reflexive. □
Problem 2. [10 points] For the following sum, find an upper and a lower bound that differ by at most 1.

\[ \sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} \]

Solution. To find the upper bound, we use the integral method, where \( f(i) = \frac{1}{\sqrt{i^3}} \):

\[
\sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} \leq f(1) + \int_{1}^{\infty} \frac{1}{\sqrt{i^3}} dx
\]

\[
= 1 - 2 \left. \frac{1}{\sqrt{i}} \right|_{1}^{\infty}
\]

\[
= 1 - 2 (0 - 1) = 3
\]

To find the lower bound, we use also use the integral method:

\[
\sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} \geq \lim_{x \to \infty} \frac{1}{\sqrt{i^3}} + \int_{1}^{\infty} \frac{1}{\sqrt{i^3}} dx
\]

\[
= 0 + 2 = 2
\]

The two bounds differ by exactly 1. We conclude that

\[
2 \leq \sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} \leq 3.
\]
Problem 3. [10 points] State whether each of the following claims is True or False and prove your answer.

(a) [2 pts] \( x \ln x \) is \( O(x) \)

**Solution.** False. \( \lim_{x \to \infty} \frac{x \ln x}{x} = \lim_{x \to \infty} \ln x = \infty > 0 \)

(b) [2 pts] \( x/100 \) is \( o(x) \)

**Solution.** False. In this case we have \( \frac{x/100}{x} = \frac{1}{100} \to 1/100 > 0 \) as \( x \to \infty \)

(c) [2 pts] \( x^{n+1} \) is \( \Omega(x^n) \)

**Solution.** True. Taking the quotient we arrive to \( \frac{x^{n+1}}{x^n} = x \to \infty > 0 \)

(d) [4 pts] \( n! \) is \( \Theta(n^n) \).

**Solution.** False. Stirling’s formula asserts
\[
n! \sim \frac{n^n \sqrt{2\pi n}}{e^n} \quad \text{so} \quad \lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \frac{n^n \sqrt{2\pi n}}{n^n e^n} = \lim_{n \to \infty} \frac{\sqrt{2\pi n}}{e^n} = 0.
\]
Hence \( n! \) is not \( \Theta(n^n) \).

Problem 4. [20 points]

We define the following recurrence for \( n \geq 0 \):
\[
T_{n+2} = T_{n+1} + 2T_n
\]

where \( T_0 = T_1 = 1 \).

(a) [8 pts] Prove by induction that \( T_n \) is odd for \( n \geq 0 \). You do not need to solve the recurrence for this.

**Solution.** We use induction on \( n \). Let \( P(n) \) be the proposition that \( T_n \) is odd.

*Base case: \( P(0) \) is true because \( T_0 = 1 \), an odd number.*

*Inductive step: Assume that \( P(n) \) is true where \( n \geq 0 \); that is, \( T_n \) is odd. We must show that \( T_{n+1} \) is odd. But we know that \( T_{n+1} = T_n + 2T_{n-1} \). And by our induction hypothesis \( P(n) \) is true, so \( T_n \) is odd. But we now have that \( T_{n+1} \) is a sum of an odd number, \( T_n \) and an even number \( 2T_{n-1} \), and thus must be odd as well.*

The theorem follows by induction.
(b) [12 pts] Prove by induction that \( \gcd(T_{n+1}, T_n) = 1 \) for \( n \geq 0 \). You may assume that \( T_n \) is odd for all \( n \). You do not need to solve the recurrence for this.

**Solution.** We use induction on \( n \). Let \( P(n) \) be the proposition that \( \gcd(T_{n+1}, T_n) = 1 \).

**Base case:** \( P(0) \) is true because \( T_0 = 1 \) and \( T_1 = 1 \) are relatively prime.

**Inductive step:** Assume that \( P(n) \) is true where \( n \geq 0 \); that is, \( \gcd(T_{n+1}, T_n) = 1 \). We must show that \( \gcd(T_{n+2}, T_{n+1}) = 1 \). Assume not, then \( T_{n+2} \) and \( T_{n+1} \) have a common divisor \( d > 1 \). That is \( d|T_{n+2} \) and \( d|T_{n+1} \). Since all \( T_n \) are odd, we know that \( d \) cannot be even. But then \( d \) must also divide the linear combination \( T_{n+2} - T_{n+1} = T_n \), showing that \( \gcd(T_{n+1}, T_n) = d \), contradicting the assumption that \( T_n \) and \( T_{n+1} \) are relatively prime. So \( \gcd(T_{n+2}, T_{n+1}) = 1 \), as desired.

The theorem follows by induction. ■

**Problem 5. [12 points]**

Find a closed-form solution to the following recurrence:

\[
\begin{align*}
  x_0 &= 4 \\
  x_1 &= 23 \\
  x_n &= 11x_{n-1} - 30x_{n-2} \text{ for } n \geq 2.
\end{align*}
\]

**Solution.** The characteristic equation is \( r^2 - 11r - 30 = 0 \).

Factoring this out we get \((r - 5)(r - 6)\), so our roots are

\[
\begin{align*}
  r_1 &= 5 \\
  r_2 &= 6
\end{align*}
\]

Therefore a general form for a solution is

\[
  x_n = A(5)^n + B(6)^n
\]

Substituting the initial conditions into this general form gives a system of linear equations.

\[
\begin{align*}
  4 &= A + B \\
  23 &= 5A + 6B
\end{align*}
\]

The solution to this linear system is \( A = 1 \) and \( B = 3 \). The complete solution to the recurrence is therefore

\[
  x_n = 5^n + 3 \cdot 6^n
\]

■
Problem 6. [10 points] Note: in this question, you may use “choose” notation or factorials in your answers for both (a) and (b). In the card game of bridge, you are dealt a hand of 13 cards from the standard 52-card deck.

(a) [5 pts] A balanced hand is one in which a player has roughly the same number of cards in each suit. How many different hands are there where the player has 4 cards in one suit and 3 cards in each of the other suits?

Solution. There are 4 suits to pick from for the longest suit, 4 cards out of 13 to choose from in that suit, and 3 cards out of 13 to choose from in each of the remaining 3 suits. This gives

\[4 \cdot \binom{13}{4} \cdot \binom{13}{3} \cdot \binom{13}{3} \cdot \binom{13}{3}\]

such hands.

(b) [5 pts] Not surprisingly, a non-balanced hand is one in which a player has more cards in some suits than others. Hands that are very desired are ones where over half the cards are in one suit. How many different hands are there where there are exactly 7 cards in one suit?

Solution. There are 4 suits to choose from for the long suit, 7 cards out of 13 to choose in that suit, and 6 cards out of the remaining 39 in the other 3 suits. This gives

\[4 \cdot \binom{13}{7} \cdot \binom{39}{6}\]

such hands.
Problem 7. [15 points] Circle every symbol on the left that could correctly appear in the box to its right. For each of the six parts you may need to circle any number of symbols.

(a) \( O \ \Omega \ \Theta \ o \ \omega \ \sim \quad 6n^2 + 7n - 10 = O, \Omega, \Theta \left( n^2 \right) \)

(b) \( O \ \Omega \ \Theta \ o \ \omega \ \sim \quad 6^n = \Omega, \omega \left( n^6 \right) \)

(c) \( O \ \Omega \ \Theta \ o \ \omega \ \sim \quad n! = O, o \left( n^n \right) \)

(d) \( O \ \Omega \ \Theta \ o \ \omega \ \sim \quad \sum_{j=1}^{n} \frac{1}{j} = O, \Omega, \Theta, \sim (\ln n) \)

(e) \( O \ \Omega \ \Theta \ o \ \omega \ \sim \quad \ln(n^3) = O, \Omega, \Theta \left( \ln n \right) \)
Problem 8. [10 points] Give upper and lower bounds for the following expression which differ by at most 1.

\[ \sum_{i=1}^{n} \frac{1}{i^3} \]

**Solution.** To find upper and lower bounds, we use the integral method:

\[ \sum_{i=1}^{n} \frac{1}{i^3} \leq 1 + \int_{1}^{n} \frac{1}{x^3} dx \]

\[ = 1 - \frac{1}{2} x^{-2} \bigg|_{1}^{n} \]

\[ = 1 - \frac{1}{2} \left( \frac{1}{n^2} - 1 \right) = \frac{3}{2} - \frac{1}{2n^2} \]

\[ \sum_{i=1}^{n} \frac{1}{i^3} \geq \frac{1}{n^3} + \int_{1}^{n} \frac{1}{x^3} dx \]

\[ = \frac{1}{n^3} - \frac{1}{2} x^{-2} \bigg|_{1}^{n} \]

\[ = \frac{1}{n^3} - \frac{1}{2} \left( \frac{1}{n^2} - 1 \right) = \frac{1}{2} + \frac{1}{n^3} - \frac{1}{2n^2} \]

Problem 9. [10 points] Let \( T(n) \) be a recurrence such that for all integers \( n > 8 \),

\[ T(n) = 16T(\lfloor n/2 + \log n \rfloor) + n^4 \]

Assume that \( T(n) = 0 \) for \( n \leq 8 \). Find a \( \Theta \) bound for \( T(n) \). Show your work.

**Solution.** Use Akra-Bazzi: \( a_1 = 16, b_1 = 1/2, h_1(n) = \lfloor n/2 + \log n \rfloor - n/2, g(n) = n^4, p = 4, \)

\[ T(n) = \Theta \left( n^4 \left( 1 + \int_{1}^{n} \frac{1}{u^5} du \right) \right) = \Theta \left( n^4 \left( 1 + \int_{1}^{n} \frac{1}{u} du \right) \right) = \Theta(n^4 \log n) \]
Problem 10. [10 points] At the end of year 0, Karen and Joe both have no money. In each subsequent year, the following happens:

1. On November 15th, Joe, an extremely good investor, has three times the amount of money he had at the beginning of the year.

2. On December 1st, Joe gives Karen the amount of money he had at the beginning of the year.

3. On December 15th, Karen makes $10, which she promptly gives to Joe.

Find a linear recurrence for the total amount of money $T_n$ that the two have between them at the end of year $n$, including base cases. You do not have to solve the recurrence.

You may choose to define recurrences, $K_n$ and $J_n$, for the amount of money that each of Karen and Joe have, respectively, but your final answer must be solely in terms of $T_n$.

(Hint: First find an expression for the amount of money Joe has at the end of year $n - 1$, $J_{n-1}$, in terms of $T_{n-1}$ and $T_{n-2}$.)

Solution. At the end of year $n$, Karen has the amount she had at the end of year $n - 1$, plus the amount that Joe had at the beginning of year $n$ (which is the same as the amount he had at the end of year $n - 1$). Hence, $K_n = K_{n-1} + J_{n-1}$.

At the end of year $n$, Joe has three times the amount he had at the beginning of the year, minus the amount he had at the beginning of the year, plus the $10 Karen gave him. Hence, $J_n = 3J_{n-1} - J_{n-1} + 10 = 2J_{n-1} + 10$.

Now to determine the total that the two have, $T_n = K_n + J_n$. According to the hint, we rearrange the equation as $J_n = T_n - K_n$. Notice that $K_n = T_{n-1} = K_{n-1} + J_{n-1}$. Hence, we can substitute for $K_n$ to get the equation $J_n = T_n - T_{n-1}$.

Finally, we substitute back into the equation for Joe to get:

\[
J_n = 2J_{n-1} + 10
\]
\[
T_n - T_{n-1} = 2(T_{n-1} - T_{n-2}) + 10
\]
\[
T_n = 3T_{n-1} - 2T_{n-2} + 10
\]

The base cases are $T_0 = 0$ and $T_1 = 10$.

Note that this is one possible recurrence for $T_n$; other equivalent recurrences are possible. ■
Problem 11. [10 points]
Let \( T(n) \) be defined by the recurrence
\[
T(n) = 2 \sqrt{T(n-1)T(n-2)}
\]
for \( n \geq 2 \) with \( T(0) = T(1) = 1 \). Prove by induction that \( T(n) = O(2^{2n/3}) \).

Let \( P(n) \) be the proposition that \( T(n) \leq c \cdot 2^{2n/3} \), where \( c \) is a very large constant, say 100.

**Base cases:** \( T(0) = 1 \leq c \), \( T(1) = 1 \leq c \cdot 2^{2/3} \).

**Inductive step:** Assume \( P(k) \) for \( 0 \leq k \leq n \) in order to prove \( P(n+1) \). We derive
\[
T(n+1) = 2 \sqrt{T(n)T(n-1)} \\
\leq 2 \sqrt{c \cdot 2^{2n/3} \cdot c \cdot 2^{2(n-1)/3}} \quad \text{(By \( P(n) \) and \( P(n-1) \).)} \\
= c \cdot 2^{1+(2n/3+2n/3-2/3)/2} \\
= c \cdot 2^{(n+1)/3}.
\]
This proves \( P(n+1) \). □

Problem 12. [10 points] Solve the recurrence \( T(n) = T(n-1) + 12T(n-2) \) for \( n \geq 2 \) with \( T(0) = 2 \) and \( T(1) = 1 \).

Solution. The characteristic polynomial is \( x^2 = x + 12 \) with solutions \( x = 4 \) and \( x = -3 \). This gives \( T(n) = c_1 4^n + c_2 (-3)^n \) for \( n \geq 0 \).

Plugging in the base cases, we get the system of equations
\[
2 = c_1 + c_2 \\
1 = 4c_1 - 3c_2
\]
Hence, \( c_1 = c_2 = 1 \), so \( T(n) = 4^n + (-3)^n \). □
Problem 13. [10 points]
How many length $2n$ sequences of red and green balls with exactly $m$ red balls satisfy the constraint that every red ball has a green ball adjacent to it on its left? You may assume that $m \leq n$. Express your answer using a single binomial coefficient; that is, using a single term of the form $\binom{x}{y}$.

Solution. Let $B$ be the set of length $2n$ sequences with $m$ red and $2n - m$ green balls such that every red ball has a green ball on its left. Let $A$ be the set of length $2n - m$ bit sequences with exactly $m$ ones and $2n - 2m$ zeroes. Map a sequence from $A$ to a sequence in $B$ by mapping each 0 to a green ball and each 1 to a green ball followed by a red ball. This mapping is a bijection. By the bijection rule, $|B| = |A| = \binom{2n - m}{m}$.

Problem 14. [10 points]
Find a combinatorial proof of the following identity by counting the number of pairs of sets $(X, Y)$ such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$:

$$\binom{n}{m} 2^m = \sum_{i=0}^{m} \binom{n}{i} \binom{n-i}{m-i}$$

Solution. Let’s consider each side of the equation.

Looking at the left side:
One way to count the number of pairs of sets $(X, Y)$ is to first count the number of possible sets $Y$, and then count the number of possible sets $X$ for each $Y$.

The number of sets $Y \subseteq \{1, 2, \ldots, n\}$ with $|Y| = m$ is equal to $\binom{n}{m}$. For each $Y$, every value in $Y$ can either be in $X$ or not be in $X$. Since $|Y| = m$, the number of sets $X$ such that $X \subseteq Y$ is equal to the number of binary bit strings of length $m$, which equals $2^m$.

By the generalized product rule, the number of pairs of sets $(X, Y)$ such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$ is equal to $\binom{n}{m} 2^m$.

Looking at the right side:
Another way to count the number of pairs of sets $(X, Y)$ is to first count the number of possible sets $X$, and then count the number of possible sets $Y$ for each $X$.

Let $|X| = i$, where $i$ must range from 0 to $m$, since $X \subseteq Y$. For each value $i$, the number of sets $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = i$ is equal to $\binom{n}{i}$. For each $X$, we must pick the remaining $m - i$ elements from the remaining $n - i$ possible values to complete the set $Y$. Therefore, the number of sets $Y$ such that $X \subseteq Y$ is equal to $\binom{n-i}{m-i}$.

By the generalized product rule, the number of pairs of sets $(X, Y)$ such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$ is also equal to $\sum_{i=0}^{m} \binom{n}{i} \binom{n-i}{m-i}$.

\[\Box\]
Problem 15. [15 points]

Your answers for the following questions may contain binomial coefficients of the form $\binom{x}{y}$, factorials, additions, multiplications, and divisions. Please explain your terms for partial credit.

(a) [5 pts] How many 8-digit decimal sequences satisfy the following constraints?

1. Exactly four of the possible digits (the numbers from 0 to 9) appear.
2. One of the digits appears exactly five times.

For example, 01921111 is such a sequence.

Solution. If one of the four digits appears exactly five times, then each of the other three digits must appear only once each (such as in 1111234). There are 10 choices for the repeated digit and $\binom{9}{3}$ choices for the remaining digits. By the bookkeeper rule, these digits may be arranged in $\frac{8!}{5!1!1!1!}$ ways. Thus, the number of such sequences is $10 \cdot \binom{9}{3} \cdot \frac{8!}{5!1!1!1!} = 282,240$.

(b) [10 pts] How many 8-digit decimal sequences satisfy the following constraints?

1. Exactly four of the possible digits appear.
2. One of the digits appears at least four times.

For example, 01921111 and 01921112 are both such sequences.

Solution. If one of the four digits appears at least four times, then it must be the case that either one digit appears five times as in the previous part, or one digit appears exactly four times, a second digit appears exactly twice, and each of the other two digits appears only once each (such as in 1112234).

In the new case, there are 10 choices for the digit that appears four times, 9 remaining choices for the digit that appears twice, and $\binom{8}{2}$ remaining choices for the other digits. By the bookkeeper rule, these digits may be arranged in $\frac{8!}{4!2!1!1!}$ ways. Thus, there are $10 \cdot 9 \cdot \binom{8}{2} \cdot \frac{8!}{4!2!1!1!}$ such sequences.

Therefore, the total number of sequences is the sum of the two disjoint sets, which equals $10 \cdot 9 \cdot \binom{8}{2} \cdot \frac{8!}{4!2!1!1!} + 10 \cdot \binom{9}{3} \cdot \frac{8!}{5!1!1!1!} = 2,399,040$. 