Quiz 2 Solutions

- The quiz is **closed book**, but you may have two 8.5” × 11” sheet with notes (either printed or in your own handwriting) on both sides.

- Calculators and electronic devices (including cell phones) are not allowed.

- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.

- Please show your work. Partial credit cannot be given for a wrong answer if your work isn’t shown.

- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem’s page.

- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.

- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

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TA: ________________________________

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Problem 1. [11 points]

Fix positive integers $k$ and $n$ with $k \leq n$. Let

\[ \mathcal{X} = \{ S \subseteq \{1, 2, \ldots, n\} : |S| \leq k \}. \]

Consider the following relation from $\mathcal{X}$ to $\mathcal{X}$:

\[ R = \{ (S, T) \in \mathcal{X} \times \mathcal{X} : S \subseteq T \}. \]

(a) [7 pts] Prove that $R$ is a (weak) partial order on $\mathcal{X}$.

Solution. To prove that a relation is a weak partial order, we need to prove that it is reflexive, antisymmetric, and transitive.

Reflexivity: Every set $S$ is a subset of itself, that is, $S \subseteq S$.

Antisymmetry: For sets $S$ and $T$, if both $S \subseteq T$ and $T \subseteq S$, then they must contain the same integers, and therefore $S = T$.

Transitivity: If $S \subseteq T$ and $T \subseteq R$, then $S$ is necessarily a subset of $R$, that is, $S \subseteq R$. ■
(b) [4 pts] Take $n = 4$ and $k = 2$. Draw the Hasse diagram of the partial order.

Solution.
Problem 2. [10 points]

(a) [5 pts] Order the following functions from 1 through 6 so that your $i^{th}$ choice is $\text{big-O}$ of your $(i + 1)^{th}$ choice (so the function you mark 1 should be smallest, and 6 largest).

\[
\begin{align*}
n^n & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 6 \\
(\log n)^2 & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 1 \\
n^{1.0001} & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 4 \\
(1.0001)^n & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 5 \\
2^{\log_2 n} & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 2 \\
n(\log n)^{1001} & \quad \underline{\quad \quad \quad \quad \quad \quad \quad \quad} \quad 3
\end{align*}
\]

Solution.
(b) [5 pts] Identify and explain the mistake in the “proof” of the following bogus claim.

False Claim.

\[ 2^n = O(1). \]  \hfill (1)

Proof. The proof is by induction on \( n \) where the induction hypothesis, \( P(n) \), is the assertion (1).

Base case: \( P(0) \) holds trivially.

Inductive step: Assume \( P(n) \) to prove \( P(n + 1) \). So there is a constant \( c > 0 \) such that
\[ 2^n \leq c \cdot 1. \]

Therefore,
\[ 2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1, \]
which implies that \( 2^{n+1} = O(1) \). That is, \( P(n + 1) \) holds, which completes the proof of the inductive step.

We conclude by induction that \( 2^n = O(1) \) for all \( n \). \hfill \Box

Solution. The predicate \( P(n) \): \( 2^n = O(1) \) is nonsensical since you can’t have \( n \) be a specific value (as specified by \( P(n) \)) and also a variable that tends to infinity as in the definition for big-Oh. So \( P(n) \): \( 2^n = O(1) \) is not a valid predicate for induction purposes. \hfill \blacksquare
Problem 3. [10 points]

(a) [3 pts] Prove that

\[ \log(n!) = \Theta(n \log n). \]

Solution.

\[
\begin{align*}
  n! & \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \\
  \log n & \sim \log \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \\
  & = \log \sqrt{2\pi n} + \log \left( \frac{n}{e} \right)^n \\
  & = \frac{1}{2} \log 2\pi + \frac{1}{2} \log n + n \log n - n \log e \\
  & = n \log n + c_1 n + c_2 \log n + c_3 \\
  & = \Theta(n \log n)
\end{align*}
\]

■
(b) [7 pts] Show that 
\[ \sum_{k=2}^{n} \frac{1}{\log(k!)} = \Theta(\log \log n). \]

**Solution.**
\[ \sum_{k=2}^{n} \frac{1}{\log(k!)} = \sum_{k=2}^{n} \frac{1}{\Theta(k \log k)} = \Theta \left( \sum_{k=2}^{n} \frac{1}{k \log k} \right) \]

Then we use the integral method for bounds. Noting that \( \frac{1}{k \log k} \) is a non-increasing function, and using \( f(x) = \frac{1}{x \log x} \) we can say
\[ \int_{2}^{n} \frac{dx}{x \log x} \leq \sum_{k=2}^{n} \frac{1}{k \log k} \leq f(2) + \int_{2}^{n} \frac{dx}{x \log x} \]
\[ \log \log x \bigg|_{x=2}^{x=n} \leq \sum_{k=2}^{n} \frac{1}{k \log k} \leq f(2) + \log \log x \bigg|_{x=2}^{x=n} \]
\[ \log \log n - f(2) \leq \sum_{k=2}^{n} \frac{1}{k \log k} \leq f(2) + \log \log n - f(2) \]
\[ \log \log n - f(2) \leq \sum_{k=2}^{n} \frac{1}{k \log k} \leq \log \log n \]

and so, by definition of \( \Theta \), we have that
\[ \sum_{k=2}^{n} \frac{1}{k \log k} = \Theta(\log \log n) \]

which means that
\[ \sum_{k=2}^{n} \frac{1}{\log k!} = \Theta(\log \log n) \]
Problem 4. [10 points]

Give a closed-form solution to the following recurrence relation:

\[ G_n = 6G_{n-1} - 9G_{n-2} + 4n \quad \text{for } n \geq 2, \quad G_0 = 2, \quad G_1 = 7. \]

Solution. We first solve for the homogeneous part of the recurrence, which is \( G_n = 6G_{n-1} - 9G_{n-2} \). Naturally, we assume that the solution is an exponential, and try \( G_n = \alpha^n \). This gives us

\[
\begin{align*}
\alpha^n &= 6\alpha^{n-1} - 9\alpha^{n-2} \\
\alpha^2 &= 6\alpha - 9 \\
\alpha^2 - 6\alpha + 9 &= 0 \\
(\alpha - 3)^2 &= 0
\end{align*}
\]

and we have our characteristic polynomial. The roots of the polynomial are 3 of multiplicity 2, so the homogeneous solution has the form

\[ G_n = c_1 \cdot 3^n + c_2 \cdot n3^n. \]

Now we move on to the non-homogeneous part \((4n)\) and find a particular solution for the recurrence. We first try \( G_n = an + b \). Substituting, we get

\[
\begin{align*}
an + b &= 6(a(n - 1) + b) - 9(a(n - 2) + b) + 4n \\
an + b &= 6an - 6a + 6b - 9an + 18a - 9b + 4n \\
4an - 12a + 4b &= 4n
\end{align*}
\]

This leads to the system of equations

\[
\begin{align*}
4a &= 4 \\
-12a + 4b &= 0
\end{align*}
\]

and we find that \( a = 1 \) and \( b = 3 \).

Adding together the homogeneous solution and the particular solution gives the general solution

\[ G_n = c_1 \cdot 3^n + c_2 \cdot n3^n + n + 3. \]

Using our initial conditions, we get the equations

\[
\begin{align*}
G_0 &= 2 = c_1 + 3 \\
G_1 &= 7 = c_1 \cdot 3 + c_2 \cdot 3 + 1 + 3
\end{align*}
\]

and we find that \( c_1 = -1 \) and \( c_2 = 2 \). Then the closed-form solution for the recurrence is

\[ G_n = -3^n + 2n3^n + n + 3. \]
Problem 5. [8 points] Find a $\Theta$ bound for the solution to the following recurrence:

$$T(n) = \begin{cases} 
1 & \text{if } n \leq 4 \\
2T\left(\left\lfloor \frac{n}{4} \right\rfloor + \sqrt{n}\right) + \sqrt{n} & \text{if } n > 4 
\end{cases}$$

Solution. We use the Akra-Bazzi formula. $p$ is the solution to $2 \left(\frac{1}{4}\right)^p = 1$, so $p = 1/2$.

\[
\Theta\left(n^{1/2} + n^{1/2} \int_1^n \frac{\sqrt{x}}{x^{3/2}} \, dx\right) = \Theta\left(n^{1/2} + n^{1/2} \int_1^n \frac{1}{x} \, dx\right) = \Theta(n^{1/2} + n^{1/2} \ln n) = \Theta(n^{1/2} \ln n)
\]
Problem 6. [10 points] Recall that the Fibonacci numbers are defined by

\[ F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3. \]

Prove, by induction, that

\[ F_{n+1}F_{n-1} = (F_n)^2 + (-1)^n \quad \text{for all } n \geq 2. \]

Solution. We define our inductive hypothesis \( P(n) \): \( F_{n+1}F_{n-1} = (F_n)^2 + (-1)^n \).

Base Case: \( P(2) \). Noting that \( F_3 = 2 \), we observe that

\[ 2 \cdot 1 = 1^2 + 1 \]

so our base case holds.

Inductive Step: We want to show that \( P(n) \Rightarrow P(n+1) \), and we assume for the purposes of induction that \( P(n) \) holds. Then

\[ F_{n+1}F_{n-1} = (F_n)^2 + (-1)^n \]

and we can also note that

\[ (F_n)^2 = F_{n+1}F_{n-1} - (-1)^n \]

Let us start with the product \( F_{n+2}F_n \). Using the Fibonacci recursion, we have

\[ F_{n+2}F_n = (F_{n+1} + F_n)F_n = F_{n+1}F_n + (F_n)^2 \]

Substituting for \( (F_n)^2 \) yields

\[ F_{n+2}F_n = F_{n+1}F_n + F_{n+1}F_{n-1} - (-1)^n \]
\[ = F_{n+1}(F_n + F_{n-1}) + (-1)(-1)^n \]
\[ = F_{n+1}F_{n+1} + (-1)^{n+1} \]
\[ = (F_{n+1})^2 + (-1)^{n+1} \]

\[ \Rightarrow P(n+1). \]
Problem 7. [12 points]

In a famous scene from the movie *Ocean's Eleven*, Brad Pitt’s character Rusty is teaching poker to some serious beginners. Shane West (playing himself) accidentally finds himself with a hand consisting of three pairs, which draws the remark from Rusty, “You can’t have three pairs. You can’t have 6 cards in a 5-card game.”

Let’s consider the fictional game of 6-card poker, played from a standard deck of 52 cards of 4 suits (♣, ♦, ♥, ♠) and 13 ranks (2-10, J, Q, K, A).

Your solutions to the following questions may be in terms of combinatorial quantities like factorials and \( \binom{n}{k} \). However, it may help you in double-checking your answer to know that three pairs should come out to be rarer than a full house.

(a) [6 pts] How many “three-pairs” hands are there? This is a hand with 2 cards of one rank, 2 cards of a second (different) rank, and 2 cards of a third (different) rank, for example, 3♥ 3♦ 5♣ 5♥ J♠ J♥.

**Solution.** We can create a bijection for three-pairs consisting of 3 ranks and 2 suits for each rank. This comes out to \( \left( \binom{13}{3} \right)^2 = 61,776. \)
(b) [6 pts] How many full house hands are there? This is a hand consisting of 3 cards of one rank, 2 cards of another (different) rank, and one single card of a third (different) rank: for example, $6\heartsuit 6\spadesuit 6\clubsuit Q\diamondsuit Q\spadesuit A\heartsuit$ is a full house.

**Solution.** We can create a bijection for full house consisting of 1 rank with 3 suits, a second rank with 2 suits, and a third rank of any suit. This comes out to $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 164,736$, which, as confirmed by our hint, is (much) larger than 61,776.
Problem 8. [7 points] Prove that any subset $C$ of $\{101, 102, \ldots, 199, 200\}$ with $|C| = 51$ contains two numbers that sum to 301.

Solution. First, we note that the pairs of numbers $\{101, 200\}$, $\{102, 199\}$, ..., $\{150, 151\}$ add up to 301. We can see that there are only 50 such pairs, so we can guess that with 51 numbers in $C$, one of these pairs must exist in $C$.

From here, we can define our $\mathcal{A}$ [pigeons] to be the elements of $C$, and our $\mathcal{B}$ [pigeonholes] to be the sets $\{101, 200\}$, $\{102, 199\}$, ..., $\{150, 151\}$. We put a $\mathcal{A}$ (a number/element of $C$) into a $\mathcal{B}$ (a pair of numbers) if the element is one of the elements in the set.

\[
\mathcal{A} \to \mathcal{B} \mathcal{B}
\]

Since we have 51 $\mathcal{A}$ and 50 $\mathcal{B}$, at least one of the $\mathcal{B}$ must have two $\mathcal{A}$, which means at least one pair of numbers is in $C$ which sums to 301.

\[\blacksquare\]
Problem 9. [12 points] Consider an $n \times n$ chessboard (in the figure below, $n = 8$).

(a) [4 pts] How many ways are there to place $n$ rooks on the board, such that no two rooks are in the same row or the same column?

Solution. For the first column, you have $n$ choices for where to put the first rook, then $n - 1$ choices for the second column, and so on. This is simply $n!$. ■
(b) [8 pts] Now suppose that the 2 squares on the diagonal in the top-left corner (as shown in gray) cannot be used. How many ways are there to place $n$ rooks so that no rook is placed on any of these gray squares, and again no two rooks are in the same row or column?

**Solution.** The set of solutions with this restriction is exactly equal to the set of all solutions to the previous part (no restrictions) minus the solutions that have rooks in either or both of the two grayed positions.

We use the inclusion-exclusion principle: The answer is the total number of configurations with no restrictions ($n!$), minus the number of configurations where there is a rook in square $(1,1)$, minus the number of configurations where there is a rook in square $(2,2)$, plus the number of configurations where there are rooks in both squares. This is

$$n! - 2(n-1)! + (n-2)!$$

Correct solutions where the rooks were considered to be distinguishable were given full credit; in that case, the answers to a) and b) are both larger by a factor of $n!$. ■
Problem 10. [10 points] Give a combinatorial proof of the identity

\[ \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n \]

for all positive integers \( n \).

(You will get partial credit (up to a maximum of half the points) for a proof that is not a combinatorial proof).

Solution. We should be able to recognize that \( 3^n \) signifies the number of ways to count the number of \( n \)-length ternary sequences; that is, a sequence of 3 distinct symbols of total length \( n \), with no restrictions on how many of each symbol we have. (Without loss of generality, we can just use 0, 1, and 2 as the symbols.) Then, for each element of the sequence, there are three choices, so we have \( 3^n \) total sequences.

Another way of counting the number of \( n \)-length ternary sequences is to count the number of sequences where the total number of 0’s and 1’s is \( k \) (not \( k \) each, but that the combined amount is \( k \)). There are \( \binom{n}{k} \) ways to place the \( k \) symbols, and we have 2 choices for each position, so this is \( 2^k \binom{n}{k} \) where \( k \) runs from 0 up to \( n \). The remaining positions of the sequence are filled with 2’s, so this does not affect the equation.

Therefore, since we are counting the number of \( n \)-length ternary sequences in two different ways, we conclude that

\[ \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n \]

. 