Practice Quiz 1 Solutions

Problem 1. [10 points]

Consider these two propositions:

\[ P: (A \lor B) \Rightarrow C \]
\[ Q: (\neg C \Rightarrow \neg A) \lor (\neg C \Rightarrow \neg B) \]

Which of the following best describes the relationship between \( P \) and \( Q \)? Please circle exactly one answer.

1. \( P \) and \( Q \) are equivalent
2. \( P \Rightarrow Q \)
3. \( Q \Rightarrow P \)
4. All of the above
5. None of the above

Draw a truth table to illustrate your reasoning. You can use as many columns as you need.

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Solution. Observe from the last two columns of the table that \( P \Rightarrow Q \) is always true, but \( Q \Rightarrow P \) is not always true (e.g. line 4). Thus \( P \) and \( Q \) are not equivalent but \( P \Rightarrow Q \).
Problem 2. [10 points]
Let $G_0 = 1, G_1 = 3, G_2 = 9,$ and define

$$G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \quad \text{(1)}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

Solution. The proof is by strong induction with hypothesis $P(n) := G_n \leq 3^n$.

Proof. Base Cases
$n = 0$: $G_0 = 1 = 3^0$.
$n = 1$: $G_1 = 3 \leq 3^1$.
$n = 2$: $G_2 = 9 \leq 3^2$.

Inductive Step: Assume $n \geq 3$ and $P(k)$ for all $k$ such that $0 \leq k \leq n$.

$$
G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \quad \text{by (1)}
\leq 3^{n-1} + (3)3^{n-2} + (3)3^{n-3} \quad \text{by induction hypothesis}
= 3^{n-2}[3 + 3 + 1]
= (7)3^{n-2}
\leq (9)3^{n-2}
= 3^n
\Box
$$
Problem 3. [20 points]

In the game of Squares and Circles, the players (you and your computer) start with a shared finite collection of shapes: some circles and some squares. Players take turns making moves. On each move, a player chooses any two shapes from the collection. These two are replaced with a single one according to the following rule:

A pair of identical shapes is replaced with a square. A pair of different shapes is replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [5 pts] Prove that the game will end.

Solution. Proof. We use induction on the number of turns to prove the statement. Let \( n \) be the number of shapes originally, and let \( P(k) \) be the proposition that if \( 0 \leq k \leq n-1 \) then after \( k \) turns, the number of remaining shapes is \( n-k \). Thus the game ends after \( n-1 \) steps.

Base case: \( P(0) \) is true by definition; the number of remaining shapes after 0 turns is \( n - 0 = n \), the original number of shapes.

Inductive step: Now we must show that \( P(k) \) implies \( P(k+1) \) for all \( k \geq 0 \). If \( k \geq n-1 \), \( P(k) \) implies \( P(k+1) \) would always be true, since \( P(k+1) \) would be trivially true. So we only need to prove this for \( k < n-1 \). So assume for your inductive hypothesis that \( P(k) \) is true, where \( 0 \leq k < n-1 \); that is, after \( k \) turns the number of remaining shapes in \( n-k \). Since \( k < n-1 \), the number of remaining shapes is \( n-k > 1 \). Hence there are at least 2 shapes to choose from and the game has not ended yet. In the \( k+1 \)st turn either the computer will choose 2 shapes, or you will choose two shapes. In either case the two shapes chosen, will be replaced by exactly once. Hence the number of shapes remaining will be \( n - k - 2 + 1 = n - k - 1 = n - (k + 1) \) as desired. This proves that \( P(k) \) implies \( P(k+1) \) for all \( k \geq 0 \).

By the principle of induction, \( P(k) \) is true for all \( k \geq 0 \).

Hence, by our inductive hypothesis after \( n-1 \) turns, 1 shape remains, which by the problem definition implies the game ends.
(b) [15 pts] Prove that you will win if and only if the number of circles initially is odd.

**Solution.** *Proof.* We use induction on the number of turns to prove the statement. Let \( a \) be the number of circles initially, and let \( P(k) \) be the proposition that if \( 0 \leq k \leq n - 1 \) then after \( k \) turns, the number of remaining circles is \( a - 2i \), for some nonnegative integer \( i \). Thus if \( a \) is odd initially, at turn \( n - 1 \), when the game ends, \( a - 2i \) circles - an odd number - remain, and since there is only one shape remaining, there must be exactly 1 circle left, and you win.

**Base case:** \( P(0) \) is true by definition; the number of remaining circles after 0 turns is \( a - 2 \times 0 = a \), the original number of shapes.

**Inductive step:** Now we must show that \( P(k) \) implies \( P(k+1) \) for all \( k \geq 0 \). If \( k \geq n - 1 \), \( P(k) \) implies \( P(k+1) \) would always be true, since \( P(k+1) \) would be trivially true. So we only need to prove this for \( k < n - 1 \). So assume for your inductive hypothesis that \( P(k) \) is true, where \( 0 \leq k < n - 1 \); that is, after \( k \) turns the number of remaining circles is \( a - 2i_1 \), for some nonnegative integer \( i_1 \). Since \( k < n - 1 \), the number of remaining shapes is \( n - k > 1 \) (from part a), hence there are at least 2 shapes to choose from and the game has not ended yet. In the \( k+1 \)st turn either the computer will choose 2 shapes, or you will choose two shapes. If the two shapes chosen are both squares, then they are replaced by a square, and the number of circles does not change, and hence is still \( a - 2i_1 \). If the two shapes chosen are both circles, then they are replaced by a square, and the number of circles gets decreased by 2, and is \( a - 2i - 2 = a - 2(i + 1) \). If one of the shapes chosen was a circle and the other was a square, they get replaced by a circle, and again the number of circles does not change and remains \( a - 2i \). Hence in all three transitions \( P(k+1) \) holds.

By the principle of induction, \( P(k) \) is true for all \( k \geq 0 \).

Hence, by induction when the game ends the parity of the number of circles is the same as the original parity of the number of circles. So you will win only if the number of circles to begin with was odd.
Problem 4. [15 points]

(a) [8 pts] Find a number \( x \in \{0, 1, \ldots, 112\} \) such that \( 11x \equiv 1 \pmod{113} \).

**Solution.** We can do this using the pulverizer. Specifically, if we find a pair \((s, t)\) such that \( 11s + 113t = 1 \), then we know that \( 11s \equiv 1 \pmod{113} \).

\[
\begin{array}{ccc}
 x & y & \text{rem}(x, y) = x - q \cdot y \\
 113 & 11 & 3 = 113 - 10 \cdot 11 \\
 11 & 3 & 2 = 11 - 3 \cdot 3 \\
 & & = 11 - 3 \cdot (113 - 10 \cdot 11) \\
 & & = -3 \cdot 113 + 31 \cdot 11 \\
 3 & 2 & 1 = 3 - 2 \\
 & & = (113 - 10 \cdot 11) - (-3 \cdot 113 + 31 \cdot 18) \\
 & & = (4 \cdot 113 - 41 \cdot 11)
\end{array}
\]

From the above work we see that \( 1 = 4 \cdot 113 - 41 \cdot 11 \), and so \(-41 \cdot 11 \equiv 1 \pmod{113} \). Therefore \(-41\) is *an* inverse of 113. However, it is not the *unique* inverse of 113 in the range \( \{1, \ldots, 113\} \), which is given by \( \text{rem}(-41, 113) = 72 \). So the multiplicative inverse of 11 modulo 113 is 72.

(b) [7 pts] Find a number \( y \in \{0, 1, \ldots, 112\} \) such that \( 11^{112111} \equiv y \pmod{113} \) *(Hint: Note that 113 is a prime.)*

**Solution.** By Fermat’s Theorem, since 113 is prime and 113 and 11 are relatively prime, it must be that

\[
11 \cdot 11^{111} \equiv 11^{113-1} \equiv 1 \pmod{113},
\]

so \( x \equiv 111 \pmod{113} \) (where \( x \) is defined as in part a). As a result,

\[
11^{112111} \equiv 11^{112 \cdot 1000 + 111} \equiv 11^{112} \cdot 11^{1000} \cdot 11^{111} \equiv 1^{1000} \cdot x \equiv x \equiv 72 \pmod{113},
\]

so the answer is 72.
Problem 5. [20 points]
Consider the simple graph $G$ given in figure 1.

Figure 1: Simple graph $G$

(a) [4 pts] Give the diameter of $G$.

Solution. Recall that the diameter is the maximum of all shortest path lengths between pairs of vertices. Note that the shortest path length between $D$ and $F$ is 3, and all other pairs of non-adjacent vertices share a neighbor.

(b) [4 pts] Give a Hamiltonian Cycle on $G$.

Solution. One possible solution is $(A, F, E, C, D, B, A)$. This cycle and its reverse should constitute all possible solutions.

(c) [4 pts] Give a coloring on $G$ and show that it uses the smallest possible number of colors.

Solution. One possible 3-coloring is: $\{A, D, E\}$ red; $\{B, F\}$ green; C blue. Because there exists an odd-length cycle (e.g. $(B, D, C)$), no 2-coloring exists. Therefore the given coloring uses the least possible number of colors.

(d) [4 pts] Does $G$ have an Eulerian cycle? Justify your answer.

Solution. No. This follows from the fact that there exist vertices with odd degree; e.g. $B$. 
Now consider graph $H$, which is like $G$ but with weighted edges, in figure 2:

Figure 2: Weighted graph H

(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on $H$.

Solution. Kruskal’s alg (building up an acyclic subgraph) gives two possible orders: $((C, D), (B, C), (A, B), (E, F), (C, E))$ and $((C, D), (B, C), (E, F), (A, B), (C, E))$. Prim’s algorithm (building up a connected, acyclic subgraph) gives one possible order: $((C, D), (B, C), (A, B), (C, E), (E, F))$. Figure 3 below gives the MST generated in any case.
Problem 6. [25 points] Let $G$ be a graph with $m$ edges, $n$ vertices, and $k$ components. Prove that $G$ contains at least $m - n + k$ cycles. (Hint: Prove this by induction on the number of edges, $m$)

Solution. The proof is by induction on $m$ with hypothesis $P(n)$: If $G$ is a graph with $n$ vertices, $m$ edges and $k$ components, then $G$ contains at least $m + k - n = c$ cycles

Proof. Base Case $m = 0$: Let $G$ be any graph with 0 edges and $n$ vertices. Then since there are no edges, each vertex is its own connected component, hence there are $k = n$ connected components. Since there are no edges there are also no cycles. Lastly we note that $m + k - n = n + 0 - n = 0$, and hence our base case.

Inductive Step Assume that $P(m)$ holds, that is any graph with $m$ edges, $n$ vertices, and $k$ components has at least $m - n + k$ cycles. We must show that $P(m+1)$ holds.

Consider an arbitrary graph $G$ with $m + 1$ edges, $n$ vertices, and $k$ components. Suppose we remove an arbitrary edge, $e$, from $G$ to obtain $G'$. This edge, $e$, was either in a cycle in $G$ or not:

Case 1: $e$ is part of a cycle in $G$
If $e$ is in a cycle in $G$ then removing it removes at least one cycle. Furthermore, removing $e$ does not disconnect the graph so the number of components remains the same. So $G'$ has $m$ edges, $n$ vertices, and $k$ components, which by the inductive hypothesis tells us it has at least $m + k - n$ cycles. But $G$ has at least one more cycle than $G'$ (since $e$ is part of a cycle). So $G$ has at least $m + k - n + 1$, or $(m + 1) + k - n$ cycles, as desired.

Case 2: $e$ is not part of a cycle in $G$
If $e$ is not part of a cycle removing it disconnects the graph of $G$, so the number of components in $G'$ is $k + 1$. So, $G'$ contains $m$ edges, $n$
vertices, and \( k + 1 \) components, so by the inductive hypothesis it contains at least \( m + (k + 1) - n \) cycles. Now since \( e \) was not part of a cycle, \( G \) and \( G' \) have the same number of cycles. So \( G \) also has at least \( m + (k + 1) - n \) cycles. Rearranging we get that \( G \) has at least \( (m + 1) + k - n \) cycles as desired.

\[
\square
\]

**Problem 7. [10 points]** In problem set 1 you showed that the \( \text{nand} \) operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator *universal*. Another universal operator is \( \text{nor} \), defined such that \( P \lor Q \equiv \neg(P \lor Q) \).

Show how to express \( P \land Q \) in terms of: \( \text{nor} \), \( P \), \( Q \), and grouping parentheses.

**Solution.** \( \neg P \lor \neg Q = (P \lor P) \land (Q \lor Q) \).  

**Problem 8. [15 points]** We define the sequence of numbers

\[
a_n = \begin{cases} 
1 & \text{if } 0 \leq n \leq 3, \\
a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4.
\end{cases}
\]

Prove that \( a_n \equiv 1 \pmod{3} \) for all \( n \geq 0 \).

**Solution.** Proof by strong induction. Let \( P(n) \) be the predicate that \( a_n \equiv 1 \pmod{3} \).

Base case: For \( 0 \leq n \leq 3 \), \( a_n = 1 \) and is therefore \( \equiv 1 \pmod{3} \).

Inductive step: For \( n \geq 4 \), assume \( P(k) \) for \( 0 \leq k \leq n \) in order to prove \( P(n + 1) \).

In particular, since each of \( a_{n-4}, a_{n-3}, a_{n-2} \) and \( a_{n-1} \) is \( \equiv 1 \pmod{3} \), their sum must be \( \equiv 4 \equiv 1 \pmod{3} \). Therefore, \( a_n \equiv 1 \pmod{3} \) and \( P(n + 1) \) holds.  

**Problem 9. [20 points]** The Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* four tiles. As shown below, there are three ways to manipulate the puzzle:

**Shift Right:** The center disc is moved one unit to the right (if there is space)

**Rotate Disc:** The four tiles in the center disc are reversed

**Shift Left:** The center disc is moved one unit to the left (if there is space)
Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

Write your proof on the next page...

**Solution.** Order the tiles from left to right in the puzzle. Define an inversion to be a pair of tiles that is out of their natural order (e.g. 4 appearing to the left of 3).

**Lemma.** Starting from the initial state there is an odd number of inversions after any number of transitions.

**Proof.** The proof is by induction. Let $P(n)$ be the proposition that starting from the initial state there is an odd number of inversions after $n$ transitions.

**Base case:** After 0 transitions, there is one inversion, so $P(0)$ holds.

**Inductive step:** Assume $P(n)$ is true. Say we have a configuration that is reachable after $n + 1$ transitions.

1. Case 1: The last transition was a shift left or shift right

   In this case, the left-to-right order of the discs does not change and thus the number of inversions remains the same as in
2. The last transition was a rotate disc.

   In this case, six pairs of disks switch order. If there were \( x \) inversions among these pairs after \( n \) transitions, there will be \( 6 - x \) inversions after the reversal. If \( x \) is odd, \( 6 - x \) is odd, so after \( n + 1 \) transitions the number of inversions is odd.

\[\square\]

Conclusion: Since all reachable states have an odd number of inversions and the goal state has an even number of inversions (specifically 0), the goal state cannot be reached.
\[\blacksquare\]
Problem 10. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range \{0, 1, \ldots, 71\}.

Solution. Since 17 and 72 = 2^3 \cdot 3^2 are relatively prime, an inverse exists and can be found by either Euler’s theorem or the Pulverizer.

Solution 1: Euler’s Theorem

\[
\phi(72) = \phi(2^3 \cdot 3^2) = \phi(2^3) \cdot \phi(3^2) = (2^3 - 2^2)(3^2 - 3^1) = 4 \cdot 6 = 24
\]

Therefore, \(17^{\phi(72)} = 17^{23}\) is an inverse of 17. To find the inverse in the range \{0, 1, \ldots, 71\} we take the remainder using the method of repeated squaring:

\[
\begin{align*}
17 &= 17 \\
17^2 &= 289 \\
&\equiv 1 \quad \text{(since 289 = 4 \cdot 72 + 1)} \\
17^4 &\equiv 1^2 = 1 \\
17^8 &\equiv 1 \\
&\ldots etc.
\end{align*}
\]

Therefore the inverse of 17 in the range \{0, 1, \ldots, 71\} is given by,

\[
17^{23} = 17^{16}17^417^217^1 \\
\equiv 1 \cdot 1 \cdot 1 \cdot 17 \\
= 17
\]

Solution 2: The Pulverizer

\[
\begin{array}{ccc}
x & y & \text{rem } xy = x - q \cdot y \\
72 & 17 & 4 = 72 - 4 \cdot 17 \\
17 & 4 & 1 = 17 - 4 \cdot 4 \\
& & = 17 - 4 \cdot (72 - 4 \cdot 17) \\
& & = 17 \cdot 17 - 4 \cdot 72 \\
4 & 1 & 0
\end{array}
\]

Since \(17^2 - 4 \cdot 72 = 1, 17^2 \equiv 1 \pmod{72}\) and so 17 is self inverse. ■
Problem 11. [15 points] Consider a graph representing the main campus buildings at MIT.

(a) [3 pts] Is this graph bipartite? Provide a brief argument for your answer.

Solution. No, there is an odd-length cycle

(b) [4 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer.

Solution. This graph does not have an Euler circuit because there are vertices with odd degree
Problem 11 continued...

Now suppose each building has separate mail collection and drop-off boxes and each collection box has a single package destined for a unique drop-off box (i.e. a permutation). We can model this as a permutation routing problem by treating the buildings as switches, attaching an input and output terminal to each of the nine buildings, and treating the existing edges as bidirectional as in the graph below:

(c) [4 pts] Give the diameter of this graph:

Solution. The diameter is 8, the length of a shortest path between the terminals for buildings 1 and 2.

(d) [4 pts] What is the max congestion of this graph? That is, in the worst case permutation, how many packages would need to pass through a single building? Provide a brief argument for your answer.

Solution. The maximum congestion is 9. Consider a permutation where all the packages on the left side are destined for drop-offs on the right side and vice versa with the building 10 package destined for building 10. In this case, all 9 packages must pass through building 10, giving a congestion of 9, which is the maximum for a graph with nine input-output pairs.
Problem 12. [10 points]

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from $u$ to $v$ or an edge from $v$ to $u$ for every distinct pair of nodes $u$ and $v$. (The nodes represent players and an edge $u \rightarrow v$ indicates that player $u$ beats player $v$.) Consider the “beats” relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold for all tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. **transitive**
   
   **Solution.** The “beats” relation is not transitive because there could exist a cycle of length 3 where $x$ beats $y$, $y$ beats $z$ and $z$ beats $x$. By the definition of a tournament, $x$ cannot then beat $y$ in such a situation. ■

2. **symmetric**
   
   **Solution.** The “beats” relation is not symmetric by the definition of a tournament: if $x$ beats $y$ then $y$ does not beat $x$. ■

3. **antisymmetric**
   
   **Solution.** The “beats” relation is antisymmetric since for any distinct players $x$ and $y$, if $x$ beats $y$ then $y$ does not beat $x$. ■

4. **reflexive**
   
   **Solution.** The “beats” relation is not reflexive since a tournament graph has no self-loops. ■
Problem 13. [20 points] An outerplanar graph is an undirected graph for which the vertices can be placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, $K_4$, is not outerplanar but any proper subgraph of $K_4$ with strictly fewer edges is outerplanar. Some examples are provided below:

![Outerplanar Graphs](image)

Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Solution. Proof. Proof by induction on the number of nodes $n$ with the induction hypothesis $P(n) =$ "every outerplanar graph with $n$ vertices is 3-colorable."

Base case: For $n = 1$ the single node graph with no edges is trivially outerplanar and 3-colorable.

Inductive step: Assume $P(n)$ holds and let $G_{n+1}$ be an outerplanar graph with $n + 1$ vertices. There must exist a vertex $v$ in $G_{n+1}$ with degree at most 2. Removing $v$ and all its incident edges leaves a subgraph $G_n$ with $n$ vertices.

Since $G_{n+1}$ could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so $G_n$ is also an outerplanar graph. $P(n)$ implies $G_n$ is 3-colorable. Therefore we can color all the vertices in $G_{n+1}$ other than $v$ using only 3 colors and since deg$(v) \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. Therefore, $G_{n+1}$ is 3-colorable and $P(n+1)$ holds.

\[\square\]