1 Build-up error

Recall a graph is connected iff there is a path between every pair of its vertices.

False Claim. If every vertex in a graph has positive degree, then the graph is connected.

(a) Prove that this Claim is indeed false by providing a counterexample.

(b) Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the first logical mistake (unjustified step) in the proof.

Proof. We prove the Claim above by induction. Let \( P(n) \) be the proposition that if every vertex in an \( n \)-vertex graph has positive degree, then the graph is connected.

**Base cases**: \( (n \leq 2) \). In a graph with 1 vertex, that vertex cannot have positive degree, so \( P(1) \) holds vacuously.

\( P(2) \) holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

**Inductive step**: We must show that \( P(n) \) implies \( P(n + 1) \) for all \( n \geq 2 \). Consider an \( n \)-vertex graph in which every vertex has positive degree. By the assumption \( P(n) \), this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex \( x \) to obtain an \( (n + 1) \)-vertex graph:

![Diagram](attachment:vertex_graph.png)

All that remains is to check that there is a path from \( x \) to every other vertex \( z \). Since \( x \) has positive degree, there is an edge from \( x \) to some other vertex, \( y \). Thus, we can
obtain a path from \( x \) to \( z \) by going from \( x \) to \( y \) and then following the path from \( y \) to \( z \). This proves \( P(n + 1) \).

By the principle of induction, \( P(n) \) is true for all \( n \geq 0 \), which proves the Claim.

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\square
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2 Euler tours

The statement of (a) in the original version was incorrect! This has been corrected below.

(a) Prove that a graph \( G \) has an Euler tour if and only if: i) every vertex of \( G \) has even degree, and ii) the subgraph obtained after removing all isolated vertices is connected. (An isolated vertex is a vertex of degree 0.)

Note that there are two directions to prove!

(b) Come up with a necessary and sufficient condition for the existence of an Euler tour in a directed graph. Adapt your proof above to prove that your condition is the right one.

3 Connectivity

Prove that any simple graph with \( n \) nodes and strictly more than \( \frac{1}{2}(n - 1)(n - 2) \) edges is connected.