Problems for Recitation 5

Graph Basics

Let $G = (V, E)$ be a graph. Here is a picture of a graph.

Recall that the elements of $V$ are called vertices, and those of $E$ are called edges. In this example the vertices are \{A, B, C, D, E, F, G\} and the edges are

$$\{A \rightarrow B, B \rightarrow D, C \rightarrow D, A \rightarrow C, E \rightarrow F, C \rightarrow E, E \rightarrow G\}.$$ 

Deleting some vertices or edges from a graph leaves a subgraph. Formally, a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where $V'$ is a nonempty subset of $V$ and $E'$ is a subset of $E$. Since a subgraph is itself a graph, the endpoints of every edge in $E'$ must be vertices in $V'$. For example, $V' = \{A, B, C, D\}$ and $E' = \{A \rightarrow B, B \rightarrow D, C \rightarrow D, A \rightarrow C\}$ forms a subgraph of $G$.

In the special case where we only remove edges incident to removed nodes, we say that $G'$ is the subgraph induced on $V'$ if $E' = \{x \rightarrow y | x, y \in V' \text{ and } x \rightarrow y \in E\}$. In other words, we keep all edges unless they are incident to a node not in $V'$. For instance, for a new set of vertices $V' = \{A, B, C, D\}$, the induced subgraph $G'$ has the set of edges $E' = \{A \rightarrow B, B \rightarrow D, C \rightarrow D, A \rightarrow C\}$.

Problem 1

A planar graph is a graph that can be drawn without any edges crossing.
1. First, show that any subgraph of a planar graph is planar.

2. Also, any planar graph has a node of degree at most 5. Now, prove by induction that any graph can be colored in at most 6 colors.

**Problem 2**

A graph $G = (V, E)$ is called bipartite if we can divide the vertex set into two parts, the “left” part and the “right” part, so that every edge has one endpoint in the left part, and one endpoint in the right part. The figure below shows an example of a bipartite graph.

For $n$ even, consider a bipartite graph with $n/2$ vertices on the left, labelled $v_1, v_2, \ldots, v_{n/2}$, and $n/2$ vertices on the right, labelled $w_1, w_2, \ldots, w_{n/2}$. Put an edge between every node on the left and node on the right, except between $v_i$ and $w_i$ for each $1 \leq i \leq n/2$ (the figure shows this graph for $n = 6$).

(a) Find an ordering of the vertices where the basic algorithm does well, and uses only 2 colors.

(b) Find an ordering where the basic algorithm does very badly, and requires $n/2$ colors.

**Problem 3**

An undirected graph $G$ has **width** $w$ if the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \ldots, v_n$$

such that each vertex $v_i$ is joined by an edge to at most $w$ preceding vertices. (Vertex $v_j$ precedes $v_i$ if $j < i$.) Use induction to prove that every graph with width at most $w$ is $(w + 1)$-colorable.

(Recall that a graph is $k$-colorable iff every vertex can be assigned one of $k$ colors so that adjacent vertices get different colors.)