Problems for Recitation 19

Routes in a flood

You want to walk from your house at $(0, 0)$ to your friend Bob’s house at $(n, n)$. Unfortunately, much of the city has been recently flooded, making some roads impassable. You and your friend were both lucky to escape! All the roads below the diagonal connecting you and your friend have been flooded.

We will answer the question: how many shortest paths are there to your friend’s house? This is quite tricky, and involves a really beautiful bijection.

(a) Let’s begin with an easier problem.

You have another, less fortunate, friend living at coordinates $(n + 1, n − 1)$ on the map. This friend (let’s call her Sandy) is trapped in her apartment, and you want to bring her some supplies. Fortunately, you have a dinghy, and can traverse both flooded and unflooded roads.

How many shortest paths are there to Sandy?
(b) We will now count the number of shortest paths to Bob that do involve at least one flooded road. This requires a very clever bijection, and is the real key to the argument.

To get you going, here is an example of the bijection; the path to Sandy is on the left, and the resulting path to Bob is on the right. Based on this, try to figure out the rule defining the bijection, and explain why it works. Hence give the formula for the number of shortest paths to Bob involve at least one flooded road.
(c) So now you can answer the original question: how many shortest paths to Bob do not involve any flooded roads?

You should be able to simplify your answer down to the simple formula $\frac{1}{n+1} \binom{2n}{n}$. These numbers are known as *Catalan numbers*.

### Counting matching brackets

Catalan numbers appear all over the place. Consider the number of possible matched brackets of length $2n$. This is just a sequence of open and closed brackets, with $n$ open brackets and $n$ close brackets, where open and closed brackets are matched. For example,

$$( ( ( ) ( ( ) ( ) ) ) )$$

is a matched bracket sequence of length 10; but

$$( ( ( ) ) ( ) ) ( ( ) )$$

is not, even though it has the right number of open and close brackets.

By finding a bijection to the number of unflooded routes to Bob, show that the number of possible matched bracket sequences of length $2n$ is again precisely the $n$’th Catalan number, $\frac{1}{n+1} \binom{2n}{n}$. 
Counting trees

The following bijection will be a little harder to find. We’re going to count a special family of trees called plane trees.

A plane tree on \( n \) nodes is a tree with exactly \( n \) nodes, with a special “root” node (drawn at the bottom). A node can have 0, 1, 2, or more children. However the order in which the children appear (from left to right) matters. Below all of the plane trees on 4 nodes are shown; note that the 2nd and 3rd tree in this list are considered different.

Show that the number of plane trees on \( n + 1 \) nodes is again the \( n \)th Catalan number \( \frac{1}{n+1} \binom{2n}{n} \), by finding a bijection to the number of unflooded routes to Bob. As a hint, below is shown two examples of the mapping given by a particular bijection \( f \). (Your friendly TA will also provide another hint if asked...)

\[
\begin{align*}
\text{Plane Trees on 4 Nodes} \\
\begin{array}{c}
\text{V} \\
\text{V} \\
\text{V} \\
\text{Y} \\
\end{array}
\end{align*}
\]