Problems for Recitation 18

The **(ordinary) generating function** for a sequence \( \langle a_0, a_1, a_2, a_3, \ldots \rangle \) is the power series:

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots
\]

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) \( \langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle \)

(b) \( \langle 1, 1, 1, 1, 1, 1, \ldots \rangle \)

(c) \( \langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle \)

(d) \( \langle 1, 0, 1, 0, 1, 0, 1, \ldots \rangle \)

(e) \( \langle 0, 0, 0, 1, 1, 1, 1, \ldots \rangle \)

(f) \( \langle 1, 3, 5, 7, 9, 11, \ldots \rangle \)
Problem 2

Suppose that:

\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \]
\[ g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \]

What sequences do the following functions generate?

(a) \( f(x) + g(x) \)

(b) \( f(x) \cdot g(x) \)

(c) \( f(x)/(1 - x) \)
Problem 3

There is a jar containing \( n \) different flavors of candy (and lots of each kind). I’d like to pick out a set of \( k \) candies.

(a) In how many different ways can this be done?

(b) Now let’s approach the same problem using generating functions. Give a closed-form generating function for the sequence \( \langle s_0, s_1, s_2, s_3, \ldots \rangle \) where \( s_k \) is the number of ways to select \( k \) candies when there is only \( n = 1 \) flavor available.

(c) Give a closed-form generating function for the sequence \( \langle t_0, t_1, t_2, t_3, \ldots \rangle \) where \( t_k \) is the number of ways to select \( k \) candies when there are \( n = 2 \) flavors.

(d) Give a closed-form generating function for the sequence \( \langle u_0, u_1, u_2, u_3, \ldots \rangle \) where \( u_k \) is the number of ways to select \( k \) candies when there are \( n \) flavors.
Problem 4

Consider the following recurrence equation:

\[
T_n = \begin{cases} 
1 & n = 0 \\
2 & n = 1 \\
2T_{n-1} + 3T_{n-2} & (n \geq 2) 
\end{cases}
\]

Let \( f(x) \) be a generating function for the sequence \( \langle T_0, T_1, T_2, T_3, \ldots \rangle \).

(a) Give a generating function in terms of \( f(x) \) for the sequence:

\( \langle 1, 2, 2T_1 + 3T_0, 2T_2 + 3T_1, 2T_3 + 3T_2, \ldots \rangle \)

(b) Form an equation in \( f(x) \) and solve to obtain a closed-form generating function for \( f(x) \).

(c) Expand the closed form for \( f(x) \) using partial fractions.

(d) Find a closed-form expression for \( T_n \) from the partial fractions expansion.