Problem Set 4

Due: Monday, October 1

Problem 1. [12 points]

(a) [3 pts] Find the chromatic number of each graph in Figure 2.

(b) [3 pts] Color the graphs based on their chromatic number.

(c) [6 pts] Also for each graph in Figure 2, state whether removing a single vertex and all edges incident with it could decrease the graph’s chromatic number.

Problem 2. [10 points] Two graphs are isomorphic if they are the same up to a relabeling of their vertices (see Definition 5.1.3 in the book). A property of a graph is said to be preserved under isomorphism if whenever $G$ has that property, every graph isomorphic to $G$ also has that property. For example, the property of having five vertices is preserved under isomorphism: if $G$ has five vertices then every graph isomorphic to $G$ also has five vertices.

(a) [5 pts] Some properties of a simple graph, $G$, are described below. Which of these properties is preserved under isomorphism?

1. $G$ has an even number of vertices.
2. None of the numerical labels of the vertices of $G$ are an even integer.
3. $G$ has a vertex of degree 3.

(b) [5 pts] Determine which graph among the four pictured in the Figure 1 are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not.

![Figure 1: Graphs](image)

**Problem 3. [15 points]** Let $G = (V, E)$ be a graph with $v$ vertices and $e$ edges.

(a) [5 pts] Let $M$ be the maximum degree of the vertices of $G$, and let $m$ be the minimum degree of the vertices of $G$. Show that:

1. $2e/v \geq m$
2. $2e/v \leq M$

(b) [5 pts] At a 6.042 ice cream study session (where the ice cream is plentiful and it helps you study too), 279 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Are the results of this study accurate?

(c) [5 pts] How many edges does $K_n$, the complete graph on $n$ vertices, have?

**Problem 4. [10 points]** Prove or disprove the following claim: for every $n \geq 3$ ($n$ boys and $n$ girls, for a total of $2n$ people), and every way of setting the preferences for $n$ boys and $n$ girls, there is a perfect matching between the boys and girls that is unstable.
Problem 5. [40 points] The are $N$ students $s_1, s_2, \ldots, s_N$ and $M$ universities $u_1, u_2, \ldots, u_M$. University $u_i$ has $n_i$ slots for students, and we’re guaranteed that $\sum_{i=1}^{M} n_i = N$. Each student ranks all universities (no ties) and each university ranks all students (no ties).

(a) [5 pts] Design an algorithm to assign students to universities with the following properties:

1. Every student is assigned to one university.
2. $\forall i$, $u_i$ gets assigned to $n_i$ students.
3. There does not exist $s_i, s_j, u_k, u_l$ where $s_i$ is assigned to $u_k$, $s_j$ is assigned to $u_l$, $s_j$ prefers $u_k$ over $u_l$, and $u_k$ prefers $s_j$ over $s_i$.
4. It is student-optimal. This means that for all possible assignments satisfying the first three properties, and the students get their top choice of university amongst these assignments.

Hint: Your algorithm will be a slight modification of the Mating Ritual. Remember to include the termination condition.

(b) [5 pts] Show that your algorithm terminates after $NM + 1$ days.

(c) [5 pts] Show that your algorithm has the following property: if during some day a university $u_j$ has at least $n_j$ applicants, then when the algorithm terminates it accepts exactly $n_j$ students.

(d) [5 pts] Show that every student is assigned to one university.

(e) [5 pts] Show that for all $i$, $u_i$ gets assigned $n_i$ students.

(f) [5 pts] Suppose that on some day a university $u_j$ has at least $n_j$ applicants. Define the rank of an applicant $s_i$ with respect to a university $u_j$ as $s_i$’s location on the university’s preference list. So, for example, $u_j$’s favorite student has rank 1. Show that the rank of $u_j$’s least favorite applicant that it says ”Maybe, ...” to cannot decrease (e.g., going from 1000 to 1005 is decreasing) on any future day. Note that $u_j$’s least favorite applicant might change from one day to the next.

(g) [5 pts] Show that there does not exit $s_i, s_j, u_k,$ and $u_l$ where $s_i$ is assigned to $u_k$, $s_j$ is assigned to $u_l$, $s_j$ prefers $u_k$ over $u_l$, and $u_k$ prefers $s_j$ over $s_i$. Note that this is analogous to a ”rogue couple” considered in lecture.

(h) [5 pts] The realm of possibility of a student is the set of all universities $u$ for which there exists some assignment satisfying the first three properties specified in this problem (i.e., the student is assigned to $u$). Of all universities in the realm of possibility of a student, we say that the student’s favorite is optimal for that student. Show that your algorithm is student-optimal (i.e., each student is assigned to his/her optimal university).
Problem 6. [13 points] Let \((s_1, s_2, ..., s_n)\) be an permutations of the number 1, 2, ..., \(n-1, n\). For instance, for \(n = 5\), one arbitrary sequence could be \((5, 3, 4, 2, 1)\). Define the graph \(G = (V, E)\) as follows:

1. \(V = \{v_1, v_2, ..., v_n\}\)
2. \(e = (v_i, v_j) \in E\) if either:
   (a) \(j = i + 1\), for \(1 \leq i \leq n - 1\)
   (b) \(i = s_k\), and \(j = s_{k+1}\) for \(1 \leq k \leq n - 1\)

Prove that this graph is 4-colorable for any \((s_1, s_2, ..., s_n)\).

Hint: First, show that the \(n\)-node graph containing \(n - 1\) edges in a line is 2-colorable.