Problem Set 11 - Revised

Due: Friday, December 7

Problem 1. [10 points] Here are seven propositions:

\[
\begin{align*}
  x_1 \lor & \ x_3 \lor \neg x_7 \\
  \neg x_5 \lor & \ x_6 \lor x_7 \\
  x_2 \lor & \neg x_4 \lor x_6 \\
  \neg x_4 \lor & \ x_5 \lor \neg x_7 \\
  x_3 \lor & \neg x_5 \lor \neg x_8 \\
  x_9 \lor & \neg x_8 \lor x_2 \\
  \neg x_3 \lor & \ x_9 \lor x_4
\end{align*}
\]

Note that:

1. Each proposition is the OR of three terms of the form \(x_i\) or the form \(\neg x_i\).
2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables \(x_1, \ldots, x_9\) independently and with equal probability.

(a) [5 pts] What is the expected number of true propositions?

(b) [5 pts] Use your answer to prove that there exists an assignment to the variables that makes all of the propositions true.

Problem 2. [30 points] A true story from World War II:

The army needs to identify soldiers with a disease called “klep”. There is a way to test blood to determine whether it came from someone with klep. The straightforward approach is to test each soldier individually. This requires \(n\) tests, where \(n\) is the number of soldiers. A better approach is the following: group the soldiers into groups of \(k\). Blend the blood samples of each group and apply the test once to each blended sample. If the group-blend doesn’t have klep, we are done with that group after one test. If the group-blend fails the test, then someone in the group has klep, and we individually test all the soldiers in the group.

Assume each soldier has klep with probability, \(p\), independently of all the other soldiers.
(a) [10 pts] What is the expected number of tests as a function of $n$, $p$, and $k$?

(b) [10 pts] How (approximately) should $k$ be chosen to minimize the expected number of tests performed, and what is the resulting expectation? (Assume that $p \ll 1/k$; so for example $(1 - p)^k \sim 1$ and $\ln(1 - p) \sim -p$).

(c) [10 pts] What fraction of the work does the grouping method expect to save over the straightforward approach in a million-strong army where 1% have klep?

Problem 3. [20 points]
In this problem, we will (hopefully) be making tons of money! Use your knowledge of probability and statistics to keep from going broke!

Suppose the stock market contains $N$ types of stocks, which can be modeled by independent random variables. Suppose furthermore that the behavior of these stocks is modeled by a double-or-nothing coin flip. That is, stock $S_i$ has half probability of doubling its value and half probability of going to 0. The stocks all cost a dollar, and you have $N$ dollars. Say you only keep these stocks for one time-step (that is, at the end of this timestep, all stocks would have doubled in value or gone to 0).

(a) [4 pts] What is your expected amount of money if you spend all your money on one stock? Your variance?

(b) [4 pts] Suppose instead you diversified your purchases and bought $N$ shares of all different stocks. What is your expected amount of money then? Your variance?

(c) [4 pts] The money that you have invested came from your financially conservative mother. As a result, your goals are much aligned with hers. Given this, which investment strategy should you take? Note that you have to invest all your money in the stock market.

(d) [4 pts] Now instead say that you make money on rolls of dice. Specifically, you play a game where you roll a standard six-sided dice, and get paid an amount (in dollars) equal to the number that comes up. What is your expected payoff? What is the variance?

(e) [4 pts] We change the rules of the game so that your payoff is the cube of the number that comes up. In that case, what is your expected payoff? What is its variance?

Problem 4. [20 points] MIT students sometimes delay laundry for a few days (to the chagrin of their roommates). Assume all random variables described below are mutually independent.

(a) [5 pts] A busy student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability $2/3$ and 2 days with probability $1/3$. Let $B$ be the number of days a busy student delays laundry. What is $E[B]$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.
(b) [5 pts] A relaxed student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let $R$ be the number of days a relaxed student delays laundry. What is $E[R]$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

(c) [5 pts] Before doing laundry, an unlucky student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let $U$ be the expected number of days an unlucky student delays laundry. What is $E[U]$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

(d) [5 pts] A student is busy with probability $1/2$, relaxed with probability $1/3$, and unlucky with probability $1/6$. Let $D$ be the number of days the student delays laundry. What is $E[D]$?

Problem 5. [20 points] Let’s consider the two-envelope problem. First, we need a mathematical model of Professor Leighton. Let $e_k$ be the probability that he puts $k$ dollars in one envelope and $2k$ in the other. Assume that a student (Carlos) picks one of the two envelopes uniformly at random.

(a) [5 pts] What is the expected amount of money in the envelope that Carlos picks?

(b) [5 pts] What is the expected amount of money in the envelope that Carlos does not pick?

(c) [5 pts] Suppose that Carlos finds $8 in the envelope he chooses. Here is a fallacious argument:

Since he picked an envelope at random, the other envelope is equally likely to contain either $4$ or $16$. Therefore, the expected amount of money in the other envelope is:

$$\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 16 = 10$$

Therefore, he should switch!

What is the probability that the other envelope contains $4$, given that Carlos sees $8$? Why is the above argument wrong?

(d) [5 pts] Compute the expected value of the amount in the other envelope, given that Carlos has found $8$ in the envelope he picked. (Hint: you might want to consider the event $E$ that the other envelope has $4$, and use the law of total expectation).