Staff Solutions to Problem Set 7

Reading:

For this pset: Chapter 9.6–9.12 Partial Orders & Equivalence Relations, Chapter 11–11.6 on Simple Graphs.

For Friday lecture: the rest of Chapter 11.7–11.10 Coloring & Connectivity

Problem 1.
This problem asks for a proof of Lemma 9.7.2 showing that every weak partial order can be represented by (is isomorphic to) a collection of sets partially ordered under set inclusion (⊆). Namely,

Lemma. Let ≤ be a weak partial order on a set, A. For any element a ∈ A, let

\[ L(a) := \{ b ∈ A \mid b ≤ a \}, \]
\[ L := \{ L(a) \mid a ∈ A \}. \]

Then the function L : A → L is an isomorphism from the ≤ relation on A, to the subset relation on L.

(a) Prove that the function L : A → L is a bijection.

Solution. By definition, L() is a surjective function onto L, so all we have to do is prove it is an injection. To prove this, suppose L((a)) = L(b). Now since a ∈ L((a)) by reflexivity, we also have a ∈ L(b). This means a ≤ b. Likewise, b ≤ a. Hence a = b, by antisymmetry.

(b) Complete the proof by showing that

\[ a ≤ b \iff L(a) ⊆ L(b) \] (1)

for all a, b ∈ A.

Solution. For the left-to-right direction, suppose a ≤ b. To prove that L(a) ⊆ L(b), suppose c ∈ L(a), which means that c ≤ a. So by transitivity, c ≤ b, which means c ∈ L(b). Hence every c ∈ L(a) is also in L(b), which proves containment.

For the right-to-left direction, suppose L(a) ⊆ L(b). But a ∈ L(a) by reflexivity, so a ∈ L(b), which means that a ≤ b.

Problem 2.
Determine which among the four graphs pictured in the Figure 1 are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, prove that it is indeed preserved under isomorphism (you only need prove one of them).
Figure 1  Which graphs are isomorphic?

Solution. $G_1$ and $G_4$ are isomorphic. In particular, the function $f : V(G_1) \to V(G_4)$ is an isomorphism, where

$f(1) = 1 \quad f(2) = 2 \quad f(3) = 3 \quad f(4) = 8 \quad f(5) = 9$
$f(6) = 10 \quad f(7) = 4 \quad f(8) = 5 \quad f(9) = 6 \quad f(10) = 7$

$G_1$ and $G_2$ are not isomorphic to $G_3$: $G_3$ has a vertex of degree four and neither $G_1$ nor $G_2$ has one.

$G_1$ and $G_2$ are not isomorphic: $G_2$ has a cycle of length four and $G_1$ does not.

There are many examples of properties preserved under graph isomorphism noted in the Chapter 11, for example, number of vertices and edges, vertex degrees, size of cycles and connectedness. See Problem 11.5 for a formal proof that isomorphisms preserves vertex degrees.

Problem 3.

Scholars through the ages have identified twenty fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly eight of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select one additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall’s theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.
Solution. Construct a bipartite graph $G$ as follows. The vertices on the left are all students and the virtues on the right are all subset of nine virtues. There is an edge between a student and a set of 9 virtues if the student already has 8 of those virtues.

Each vertex on the left has degree 12, since each student can learn one of 12 additional virtues. The vertices on the right have degree at most 9, since each set of 9 virtues has only 9 subsets of size 8. So this bipartite graph is degree-constrained, and therefore, by Lemma 11.5.6, there is a matching for the students. Thus, if each student is taught the additional virtue in the set of 9 virtues with whom he or she is matched, then every student is unique at the end of the term.

Problem 4.

Give an example of a stable matching between 3 boys and 3 girls where no person gets their first choice. Briefly explain why your matching is stable.

Solution. Call the boys 1, 2, 3 and the girls $a, b, c$. Consider the following preference list:

<table>
<thead>
<tr>
<th>choice</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>3</td>
<td>c</td>
<td>a</td>
<td>b</td>
</tr>
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<tbody>
<tr>
<td>a</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

The matching $(1, b), (2, c), (3, a)$ is stable even though no person gets their first choice.

To see the intuition behind this solution, notice first that the first choice of any boy has that boy as her last choice and vice versa. Second, notice that everyone ends up with their second choice.

Since we show a pairing where everyone has their second choice, this is stable because the only way to have a rogue pair is for a boy or girl to want their first choice, but their first choice always likes them least so will never want to leave their current partner.