Staff Solutions to In-Class Problems Week 8, Mon.

Problem 1.
A researcher analyzing data on heterosexual sexual behavior in a group of \( m \) males and \( f \) females found that within the group, the male average number of female partners was 10% larger that the female average number of male partners.

(a) Comment on the following claim. “Since we’re assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating.”

Solution. The averages won’t be the same. According to equation (11.1),

\[
\text{Avg. # male partners} = \frac{|F|}{|M|} \cdot \text{Avg. # female partners}
\]  

(1)

So the averages simply reflect the relative sizes of the male and female populations. This means that the males could truthfully report a higher average if there were more females.

Of course if the males exaggerate, then their reported average could be as large as they choose to fantasize, whatever the size of the female population.

(b) For what constant \( c \) is \( m = c \cdot f \)?

Solution. By equation (1), the men’s average number of partners is \( f/m \) times the female’s average, so \( f/m = 1.1 \) which implies \( m = (1/1.1)f \) and \( c = 10/11 \).

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be \( x(f/m) \) times the nonvirgin female average number of partners. What is \( x \)?

Solution. The male average number of partners is \( f/m \) times the female average number of partners. (According to part (b), \( f/m = 1.1 \), but this number isn’t needed here.) When virgins are excluded, the ratio of the male’s average to the females’ average will be

\[
\frac{f - .2f}{m - .05m} = \frac{.8f}{.95m} = \frac{4/5}{19/20} \cdot \frac{f}{m},
\]

so \( x = 80/95 = 16/19 \).

(d) For purposes of further research, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.

Solution. There are more females than males, so there cannot be an injective function from the females to the males.
Problem 2. (a) Prove that in every graph, there are an even number of vertices of odd degree.

*Hint:* The Handshaking Lemma 11.2.1.

**Solution.** *Proof.* Partitioning the vertices into those of even degree and those of odd degree, we know

\[
\sum_{v \in V} d(v) = \sum_{d(v) \text{ is even}} d(v) + \sum_{d(v) \text{ is odd}} d(v)
\]

By the Handshaking Lemma, the value of the lefthand side of this equation equals twice the number of edges, and so is even. The first summand on the righthand side is even since it is a sum of even values. So the second summand on the righthand side must also be even. But since it is entirely a sum of odd values, it must must contain an even number of terms. That is, there must be an even number of vertices with odd degree.

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

**Solution.** We can represent the people at the party by the vertices of a graph. If two people shake hands, then there is an edge between the corresponding vertices. So the degree of a vertex is the number of handshakes the corresponding person performed. The result in the first part of this problem now implies that there are an even number of odd-degree vertices, which translates into an even number of people who shook an odd number of hands.

(e) Call a sequence of two or more different people at the party a *handshake sequence* if, except for the last person, each person in the sequence has shaken hands with the next person in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

*Hint:* Just look at the people at the ends of handshake sequences that start with George.

**Solution.** The handshake graph between just the people at the ends of handshake sequences that start with George is a graph, so by part (b), it must have an even number of people who shake an odd number of hands.

In particular, there must be at least one other person besides George, call him Harry, who has also shaken an odd number of hands. So the handshake sequence from George that ends with Harry is what we were looking for.

Problem 3.

There are four isomorphisms between these two graphs. List them.
Solution. These are the vertex correspondences for the four isomorphisms:
1A, 2B, 3C, 4D, 5E, 6F
1A, 2B, 3D, 4C, 5F, 6E
1B, 2A, 3C, 4D, 5E, 6F
1B, 2A, 3D, 4C, 5F, 6E

Problem 4.
For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write \(ab\) as shorthand for \(a—b\).)

(a)
\[ G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\}, \quad E_1 = \{12, 23, 14, 15, 35, 45\} \]
\[ G_2 \text{ with } V_2 = \{1, 2, 3, 4, 5, 6\}, \quad E_2 = \{12, 23, 34, 14, 25, 51\} \]

Solution. Not isomorphic: \(G_2\) has a node, 2, of degree 4, but the maximum degree in \(G_1\) is 3.

(b)
\[ G_3 \text{ with } V_3 = \{1, 2, 3, 4, 5, 6\}, \quad E_3 = \{12, 23, 34, 14, 45, 56, 26\} \]
\[ G_4 \text{ with } V_4 = \{a, b, c, d, e, f\}, \quad E_4 = \{ab, bc, cd, de, ae, ef, cf\} \]

Solution. Isomorphic (two isomorphisms) with the vertex correspondences:
1f, 2c, 3d, 4e, 5a, 6b
or 1f, 2e, 3d, 4c, 5b, 6a