Staff Solutions to In-Class Problems Week 7, Mon.

Problem 1.
A 3-bit string is a string made up of 3 characters, each a 0 or a 1. Suppose you’d like to write out, in one string, all eight of the 3-bit strings in any convenient order. For example, if you wrote out the 3-bit strings in the usual order starting with 000 001 010 . . . , you could concatenate them together to get a length $3 \cdot 8 = 24$ string that started 000001010 . . . .

But you can get a shorter string containing all eight 3-bit strings by starting with 000 . . . . Now 000 is present as bits 1 through 3, and 001 is present as bits 2 through 4, and 010 is present as bits 3 through 5, . . . .

(a) Say a string 3-good if it contains every 3-bit string as 3 consecutive bits somewhere in it. Find a 3-good string of length 10, and explain why this is the minimum length for any string that is 3-good.

Solution. The string 0001110100 is a length 10 string that is 3-good. You can’t do better: there must be two bits to start and each additional bit can yield at most one new 3-bit string.

(b) Explain how any walk that includes every edge in the graph shown in Figure 1 determines a string that is 3-good. Find the walk in this graph that determines your good 3-good string from part (a).

Solution. A string can be built up from any walk by starting with the $k$ bits in the vertex at the start of the walk and successively adding the bit that labels the edge to the end of the string being built. If the walk includes every edge, then any string $b_1b_2b_3$ will appear as a substring when the edge $(b_1b_2 \rightarrow b_2b_3)$ appears in the walk.

In particular, the string 0001110100 is determined by the walk that goes through the following sequence of edges:

$$
(00 \rightarrow 00) \ (00 \rightarrow 01) \ (01 \rightarrow 11) \ (11 \rightarrow 11) \ (11 \rightarrow 10) \ (10 \rightarrow 01) \ (01 \rightarrow 10) \ (10 \rightarrow 00).
$$

(c) Explain why a walk in the graph of Figure 1 that includes every edge exactly once provides a minimum length 3-good string.

Solution. Since there are 8 edges, the string determined by the walk will be of length 10, which is minimum possible as observed in part (a). Since the walk includes every edge, it will determine a 3-good string by part (b).

(d) The situation above generalizes to $k \geq 2$. Namely, there is a digraph, $B_k$, such that $V(B_k) := \{0, 1\}^k$, and any walk through $B_k$ that contains every edge exactly once determines a minimum length $(k + 1)$-good bit-string. What is this minimum length?

Define the transitions of $B_k$. Verify that the in-degree and out-degree of every vertex is even, and that there is a positive path from any vertex to any other vertex (including itself) of length at most $k$.

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1Problem 9.11 shows that if the in-degree of every vertex of a digraph is equal to its out-degree, and there are paths between any two vertices, then there is a closed walk that includes every edge exactly once. So the graph $B_k$ implies that there always is a length-$2^k + k$ bit-string in which every length-$(k + 1)$ bit-string appears as a substring. Such strings are known as de Bruijn sequences.
Solution. A string of length $n$ has exactly $n - k$ locations where a length $k + 1$ subsequence can begin. Since there are $2^{k+1}$ length-$(k + 1)$ bit strings, the minimum length, $n$ of any $(k + 1)$ good string must satisfy $n - k \geq 2^{k+1}$, so the minimum length is at least $2^{k+1} + k$. This is exactly the length string that would be determined by a walk containing all $2 \cdot 2^k$ edges in the graph $B_k$.

\[ E(B_k) := \{ \langle a \rightarrow b \rangle \mid x \in \{0, 1\}^{k-1} \text{ AND } a, b \in \{0, 1\} \} \]

If $y \in \{0, 1\}^k$, then $y = xa$ and $y = bz$ for unique strings $x, z \in \{0, 1\}^{k-1}$ and bits $a, b \in \{0, 1\}$. Then by definition of $E(B_k)$, there are exactly two edges out of $y$, one going to $0x$ and the other to $1x$, so \text{outdeg}(y) = 2$. Likewise, there are only two edges into $y$, one from $z0$ and the other from $z1$, so \text{outdeg}(y) = 2$.

To get from vertex $b_1 b_2 \ldots b_k$ to $c_1 c_2 \ldots c_k$ with a length-$k$ walk, proceed as follows:

\[ b_1 b_2 \ldots b_k \rightarrow c_k b_1 b_2 \ldots b_{k-1} \rightarrow c_{k-1} c_k b_1 b_2 \ldots b_{k-2} \rightarrow \cdots \rightarrow c_2 c_3 \ldots c_k b_1 \rightarrow c_1 c_2 \ldots c_k \]

Problem 2.

If $a$ and $b$ are distinct nodes of a digraph, then $a$ is said to cover $b$ if there is an edge from $a$ to $b$ and every path from $a$ to $b$ includes this edge. If $a$ covers $b$, the edge from $a$ to $b$ is called a covering edge.

(a) What are the covering edges in the DAG in Figure 2?

Solution. $\langle 1 \rightarrow 2 \rangle, \langle 1 \rightarrow 3 \rangle, \langle 1 \rightarrow 5 \rangle, \langle 2 \rightarrow 4 \rangle, \langle 2 \rightarrow 6 \rangle, \langle 3 \rightarrow 6 \rangle$.

(b) Let covering ($D$) be the subgraph of $D$ consisting of only the covering edges. Suppose $D$ is a finite DAG. Explain why covering ($D$) has the same positive walk relation as $D$.

*Hint:* Consider longest paths between a pair of vertices.
Solution. What we need to show is that if there is a path in $D$ between vertices $a \neq b$, then there is a path consisting only of covering edges from $a$ to $b$. But since $D$ is a finite DAG, there must be a longest path from $a$ to $b$. Now every edge on this path must be a covering edge or it could be replaced by a path of length 2 or more, yielding a longer path from $a$ to $b$.

(c) Show that if two DAG’s have the same positive walk relation, then they have the same set of covering edges.

Solution. Proof. Suppose $C$ and $D$ are DAG’s with the same positive walk relation and that $(a \rightarrow b)$ is a covering edge of $C$. We want to show that $(a \rightarrow b)$ must also be a covering edge of $D$.

Since $(a \rightarrow b)$ itself defines a (length one) positive length path in $C$, there must be a positive length walk in $D$, and hence a positive length path, from $a$ to $b$. If this positive length path in $D$ is of length greater than one, then the path must consist of a positive length path from $a$ to $c$ followed by a positive length path from $c$ to $b$ for some vertex, $c$. Also, since $D$ is a DAG, $c$ cannot be $a$ or $b$.

This means there must also be positive length walks in $C$, and hence positive length paths, from $a$ to $c$ and from $c$ to $b$, and neither of these paths can include $(a \rightarrow b)$ or there would be a cycle. Hence the path from $a$ to $c$ to $b$ is a path in $C$ that does not include $(a \rightarrow b)$, contradicting the fact that $(a \rightarrow b)$ is a covering edge of $C$.

In sum, there is a length one path from $a$ to $b$ in $D$, namely $(a \rightarrow b)$, and this is the only path from $a$ to $b$ in $D$, which proves that $(a \rightarrow b)$ is a covering edge in $D$.

(d) Conclude that covering $(D)$ is the unique DAG with the smallest number of edges among all digraphs with the same positive walk relation as $D$.

Solution. By part (c), any DAG with the same positive walk relation as $D$ must contain all the edges of covering $(D)$. By part (b), covering $(D)$ has this same positive walk relation. It follows immediately that covering $(D)$ is the unique minimum-size DAG with the same positive walk relation as $D$.

The following examples show that the above results don’t work in general for digraphs with cycles.

STAFF NOTE: Tell students to skip these last two parts until they have done the remaining problems.

(e) Describe two graphs with vertices \{1, 2\} which have the same set of covering edges, but not the same positive walk relation (Hint: Self-loops.)

Solution. Let one graph have edges \{(1, 2), (1, 1)\} and the other \{(1, 2), (2, 2)\}. They have the same set of covering edges, namely, \{(1, 2)\}. But in the second there is a positive length path from 2 to 2, namely a path of length one but there is no positive length path from 2 to 2 in the first graph.

(f) (i) The complete digraph without self-loops on vertices 1, 2, 3 has edges between every two distinct vertices. What are its covering edges?

(ii) What are the covering edges of the graph with vertices 1, 2, 3 and edges $(1 \rightarrow 2)$, $(2 \rightarrow 3)$, $(3 \rightarrow 1)$?

(iii) What about their positive walk relations?

Solution. (i) There are no covering edges, since there is always a length two path from $a$ to $b$ that does not use the edge $(a \rightarrow b)$.

(ii) All three edges are the covering edges.
They have the same positive walk relation, namely, each vertex is connected to all the vertices, including itself, by positive length paths.

Problem 3.
Let \( R \) be a binary relation on a set \( A \). Regarding \( R \) as a digraph, let \( W^{(n)} \) denote the length-\( n \) walk relation in the digraph \( R \), that is,

\[
a \ W^{(n)} \ b \iff \text{there is a length-} n \text{ walk from } a \text{ to } b \text{ in } R.
\]

(a) Prove that

\[
W^{(n)} \circ W^{(m)} = W^{(m+n)}
\]

for all \( m, n \in \mathbb{N} \), where \( \circ \) denotes relational composition. \(^2\)

Solution. Proof. Any length-\((m + n)\) walk between vertices \( u \) and \( v \) begins with a length-\( m \) walk starting at \( u \) and ending at some vertex, \( w \), followed by a length-\( n \) walk starting at \( w \) and ending at \( v \). So

\[
u \ W^{(m+n)} \ v \ \iff \ \exists w. u \ W^{(m)} \ w \ \text{AND} \ w \ W^{(n)} \ v \ \iff \ u \ W^{(n)} \circ W^{(m)} \ v
\]

(b) Let \( R^n \) be the composition of \( R \) with itself \( n \) times for \( n \geq 0 \). So \( R^0 := \text{Id}_A \), and \( R^{n+1} := R \circ R^n \). Conclude that

\[
R^n = W^{(n)}
\]

for all \( n \in \mathbb{N} \).

\(^2\) The composition of binary relations \( R : B \rightarrow C \) with \( S : A \rightarrow B \) is the binary relation \( (R \circ S) : A \rightarrow C \) defined by the rule

\[
a \ (R \circ S) \ c \ := \ \exists b \in B. (a \ S \ b) \ \text{AND} \ (b \ R \ c).
\]
Solution. Proof. By induction on $n$ with equation $(2)$ as induction hypothesis.

Base case ($n = 0$): $R^0 = \text{Id}_A$ by definition, and $W^{(0)}$ is the length-0 walk relation which also equals \text{Id}_A by definition.

Inductive step: Suppose $(2)$ holds for some $n \geq 0$. We want to prove that

$$R^{n+1} = W^{(n+1)}.$$

We first observe that

$$R = W^{(1)}$$

by definition.

Now we have

$$R^{n+1} := R \circ R^n$$

$$= W^{(1)} \circ W^{(n)} \qquad \text{(by (3) and ind. hyp. (2))}$$

$$= W^{(n+1)} \qquad \text{(by (1))}$$

This completes the proof by induction, and we conclude that $\forall n \in \mathbb{N}, R^n = W^{(n)}$.

(c) Conclude that

$$R^+ = \bigcup_{i=1}^{n} R^i$$

where $n := |A|$ and $R^+$ is the positive length walk relation of $R$.

Solution. By Theorem 9.2.3, there is a positive length walk from vertex $u$ to a different vertex $v$ iff there is a positive length path from $u$ to $v$. Likewise, by Lemma 9.5.2, there is a positive length walk from $u$ to itself iff $u$ is on a positive length cycle. Since no path can be longer than $n - 1$ and no positive length cycle can be longer than $n$, it follows that

$$u \ R^+ \ v \quad \iff \quad \exists i \in [1, n], \text{ [there is a length } i \text{ walk from } u \text{ to } v]$$

$$\iff \ \exists i \in [1, n], \ u \ W^{(i)} \ v$$

$$\iff \ u \left( \bigcup_{i=1}^{n} W^{(i)} \right) \ v$$

$$\iff \ u \left( \bigcup_{i=1}^{n} R^i \right) \ v \quad \text{by (2)}.$$