Staff Solutions to In-Class Problems Week 4, Wed.

Problem 1.
Prove by induction:
\[ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}, \]
for all \( n > 1 \).

Solution. Proof. (By Induction). The induction hypothesis, \( P(n) \), is the inequality (1).

Base Case \((n = 2)\): The LHS of (1) in this case is \( 1 + 1/4 \) and the RHS is \( 2 - 1/2 \), and
\[
\text{LHS} = \frac{5}{4} < \frac{6}{4} = \frac{3}{2} = \text{RHS},
\]
so inequality (1) holds, and \( P(2) \) is proved.

Inductive Step: Let \( n \geq 2 \) be a nonnegative integer, and assume \( P(n) \) in order to prove \( P(n + 1) \). That is, we assume (1). Adding \( 1/(n + 1)^2 \) to both sides of this inequality yields
\[
1 + \frac{1}{4} + \cdots + \frac{1}{n^2} + \frac{1}{(n + 1)^2} < 2 - \frac{1}{n} + \frac{1}{(n + 1)^2} \\
= 2 - \left( \frac{1}{n} - \frac{1}{(n + 1)^2} \right) \\
= 2 - \frac{n^2 + 2n + 1 - n}{n(n + 1)^2} \\
= 2 - \frac{n^2 + n}{n(n + 1)^2} - \frac{1}{n(n + 1)^2} \\
= 2 - \frac{1}{n + 1} - \frac{1}{n(n + 1)^2} < 2 - \frac{1}{n + 1} \quad \text{(since \( n > 0 \)).}
\]
So we have proved \( P(n + 1) \).

Problem 2. (a) Prove by induction that a \( 2^n \times 2^n \) courtyard with a \( 1 \times 1 \) statue of Bill in any position can be covered with \( L \)-shaped tiles.

Solution. Let \( P(n) \) be the proposition that for every location of Bill in a \( 2^n \times 2^n \) courtyard, there exists a tiling of the remainder.

Base case: \( P(0) \) is true because Bill fills the whole courtyard.

Inductive step: Assume that \( P(n) \) is true for some \( n \geq 0 \); that is, for every location of Bill in a \( 2^n \times 2^n \) courtyard, there exists a tiling of the remainder. Divide the \( 2^{n+1} \times 2^{n+1} \) courtyard into four quadrants, each \( 2^n \times 2^n \). One quadrant contains Bill (\( B \) in the diagram below). Place a temporary Bill (\( X \) in the diagram) in each of the three central squares lying outside this quadrant:
Now we can tile each of the four quadrants by the induction assumption. Replacing the three temporary Bills with a single L-shaped tile completes the job. This proves that $P(n)$ implies $P(n + 1)$ for all $n \geq 0$. The theorem follows as a special case.

This proof has two nice properties. First, not only does the argument guarantee that a tiling exists, but also it gives a recursive procedure for finding such a tiling. Second, we have a stronger result: if Bill wanted a statue on the edge of the courtyard, away from the pigeons, we could accommodate him!

(b) (Discussion Question) In part (a) we saw that it can be easier to prove a stronger theorem. Does this surprise you? How would you explain this phenomenon?

**Solution.** It might seem that it ought to be harder to prove a more general theorem than a less general one, but sometimes not. For example, the more general result might actually be easier because it involves fewer assumptions, and this can help in avoiding the complications of unnecessary hypotheses.

But for an induction proof in particular, using a more general induction hypothesis means we can make a stronger assumption in the induction step —namely, we can assume a stronger $P(n)$ —which can make it easier to prove the conclusion of the induction step, namely, $P(n + 1)$.

Problem 3.
Find all possible amounts of postage that can be paid exactly using 3 and 7 cent stamps. Use induction to prove that your answer is correct.

**Solution.** Proof. We can begin by observing that the following postage amounts can be made by 3 and 7 cent stamps:

- 0 no stamps
- 3  = 3
- 6  = 3 + 3
- 7  = 7
- 9  = 3 + 3 + 3
- 10 = 3 + 7,

and these are the only amounts < 12 cents that can be paid. Now we prove that every amount $\geq 12$ can also be paid. The proof is by strong induction on $n$ with induction hypothesis

$$S(n) ::= \text{exactly } n + 12 \text{ cents postage can be paid with 3 and 7 cent stamps.}$$
**Problem 4.**
The following Lemma is true, but the proof given for it below is defective. Pinpoint exactly where the proof first makes an unjustified step and explain why it is unjustified.

**Lemma 4.1.** For any prime \( p \) and positive integers \( n, x_1, x_2, \ldots, x_n \), if \( p \mid x_1x_2\ldots x_n \), then \( p \mid x_i \) for some \( 1 \leq i \leq n \).

**Bogus proof.** Proof by strong induction on \( n \). The induction hypothesis, \( P(n) \), is that Lemma holds for \( n \).

Base case \( n = 1 \): When \( n = 1 \), we have \( p \mid x_1 \), therefore we can let \( i = 1 \) and conclude \( p \mid x_i \).

Induction step: Now assuming the claim holds for all \( k \leq n \), we must prove it for \( n + 1 \).

So suppose \( p \mid x_1x_2\ldots x_{n+1} \). Let \( y_n = x_nx_{n+1} \), so \( x_1x_2\ldots x_{n+1} = x_1x_2\ldots x_{n-1}y_n \). Since the right-hand side of this equality is a product of \( n \) terms, we have by induction that \( p \) divides one of them. If \( p \mid x_i \) for some \( i < n \), then we have the desired \( i \). Otherwise \( p \mid y_n \). But since \( y_n \) is a product of the two terms \( x_n, x_{n+1} \), we have by strong induction that \( p \) divides one of them. So in this case \( p \mid x_i \) for \( i = n \) or \( i = n + 1 \).

**Solution.** Notice that nowhere in the proof is the fact that \( p \) is prime used. So if this proof were correct, the Lemma would hold not just for prime \( p \), but for any positive integer \( p \). But of course, the Lemma is false when \( p \) is not prime, for example if \( p = 6, x_1 = 3 \) and \( x_2 = 4 \), we have \( p \mid x_1x_2 \) but NOT \( p \mid x_1 \) and NOT \( p \mid x_2 \). So there has to be something wrong somewhere.

The statement “we have by strong induction that \( p \) divides one of them” is the place where the proof breaks down: it appeals to strong induction to justify applying the induction hypothesis for \( 2 = k \leq n \). But the base case was \( n = 1 \), so we can’t assume \( 2 \leq n \). Note that the reasoning above is fine for every \( n \geq 2 \), so the whole proof would be fine if we had an argument to prove the claim for \( n + 1 = 2 \).

Now in fact, if a prime, \( p \) divides \( x_1x_2 \), it must divide \( x_1 \) or \( x_2 \); this fact is obvious if we assume the uniqueness of prime factorizations of integers, but the proof here never made use of this fact. An elementary proof of this fact appears in the chapter on number theory.

Notice that uniqueness of prime factorization is a much more general result than the simple Lemma here. This Lemma is even needed in the usual proof about prime factorization, so appealing to it to prove this Lemma would be circular.