Staff Solutions to In-Class Problems Week 2, Fri.

Problem 1.
For each of the logical formulas, indicate whether or not it is true when the domain of discourse is \( \mathbb{N} \) (the nonnegative integers 0, 1, 2, \ldots), \( \mathbb{Z} \) (the integers), \( \mathbb{Q} \) (the rationals), \( \mathbb{R} \) (the real numbers), and \( \mathbb{C} \) (the complex numbers). Add a brief explanation to the few cases that merit one.

\[
\begin{align*}
\exists x. x^2 &= 2 \\
\forall x. \exists y. x^2 &= y \\
\forall y. \exists x. x^2 &= y \\
\forall x \neq 0. \exists y. xy &= 1 \\
\exists x. \exists y. x + 2y &= 2 \text{ and } 2x + 4y &= 5
\end{align*}
\]

STAFF NOTE: The few brief explanations for entries below are sufficient.

 intervene if teams start to go overboard with adding explanations (unlikely). After the problem has been team-approved (team check on their board), you can challenge a team member to provide an omitted explanation. If they had sufficient explanations (common), I like to challenge a team member with a “meta”-question, “Which was the hardest entry to fill in, and why?”

Solution.

<table>
<thead>
<tr>
<th>Statement</th>
<th>( \mathbb{N} )</th>
<th>( \mathbb{Z} )</th>
<th>( \mathbb{Q} )</th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists x. x^2 = 2 )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>(( x = \sqrt{2} ))</td>
</tr>
<tr>
<td>( \forall x. \exists y. x^2 = y )</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>(( y = x^2 ))</td>
</tr>
<tr>
<td>( \forall y. \exists x. x^2 = y )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>(take ( y &lt; 0 ))</td>
</tr>
<tr>
<td>( \forall x \neq 0. \exists y. xy = 1 )</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>(( y = 1/x ))</td>
</tr>
<tr>
<td>( \exists x. \exists y. x + 2y = 2 \text{ and } 2x + 4y = 5 )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Problem 2.
The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: \( \lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots \) (Here \( \lambda \) denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including \( = \)), variables, and the binary symbols 0, 1 denoting 0, 1.

A string like \( 01x0y \) of binary symbols and variables denotes the concatenation of the symbols and the binary strings represented by the variables. For example, if the value of \( x \) is \( 011 \) and the value of \( y \) is \( 1111 \), then the value of \( 01x0y \) is the binary string \( 0101101111 \).

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).

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<table>
<thead>
<tr>
<th>Meaning</th>
<th>Formula</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ is a prefix of $y$</td>
<td>$\exists z \ (xz = y)$</td>
<td>PREFIX($x, y$)</td>
</tr>
<tr>
<td>$x$ is a substring of $y$</td>
<td>$\exists u \exists v \ (uxv = y)$</td>
<td>SUBSTRING($x, y$)</td>
</tr>
<tr>
<td>$x$ is empty or a string of $0$'s</td>
<td>NOT(SUBSTRING(1, $x$))</td>
<td>NO-1S($x$)</td>
</tr>
</tbody>
</table>

(a) $x$ consists of three copies of some string.

**Solution.** $\exists y \ (x = yyy)$

(b) $x$ is an even-length string of $0$'s.

**Solution.** NO-1S($x$) AND $\exists y \ (x = yy)$

Some students mentioned $\lambda$ in their formulas. Technically, this is not allowed, so they need to justify it by giving a formula that means “$x = \lambda$.” This is easy, for example: $x = xx$.

A serious mistake was to try writing a recursive definition of a predicate calculus formula, as in

$$P(x) := x = \lambda \text{ OR } \exists y. \ x = 00y \text{ AND } P(y). \quad (1)$$

Such recursive formulas are, by definition, *not* part of predicate calculus—with good reason. Definition 1 resembles a simple recursive definition of a *procedure* to test if $x$ is an even length string of $0$’s, and its meaning might be explained in procedural terms. But it’s hard to figure out in general what recursively defined formulas mean. For example, let $n$ be an integer-valued variable, and suppose we tried to define a formula, $Q(n)$, that means $n$ is positive:

$$Q(n) := (n = 0 \text{ OR NOT}(Q(n + 1))) \text{ AND } (n = 1 \text{ OR } Q(n - 1)).$$

might succeed in giving a procedural explanation for this example,

(c) $x$ does not contain both a 0 and a 1.

**Solution.**

NOT[SUBSTRING(0, $x$) AND SUBSTRING(1, $x$)]

(d) $x$ is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

**Solution.** ($x = 10$) OR ($\exists y \ (x = 1y1 \text{ AND NO-1S(y)})$)

(e) An elegant, slightly trickier way to define NO-1S($x$) is:

$$\text{PREFIX}(x, 0x). \quad (*)$$

**Solution.** Prefixing $x$ with 0 rightshifts all the bits. So the $n$th symbol of $x$ shifts into the $(n + 1)$st symbol of $0x$. Now for $x$ to be a prefix of $0x$, the $n + 1$st symbol of $0x$ must match the $(n + 1)$st symbol of $x$. So if $x$ satisfies (*), the $n$th and $(n + 1)$st symbols of $x$ must match. This holds for all $n > 0$ up to the length of $x$, that is, *all* the symbols of $x$ must be the same. In addition, if $x \neq \lambda$, it must start with 0. Therefore, if $x$ satisfies (*), all its symbols must be $0$’s.

Note that it’s easy to see, conversely, that if $x = \lambda$ or $x$ is all $0$’s, then of course it satisfies (*).
**STAFF NOTE:** Explain why can’t we define “x is an even-length string of 0’s,” by

\[ \text{PREFIX}(x, 0 \ 0 \ x). \] (**)

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**Problem 3.**

Provide a counter model for the invalid implication. Informally explain why the other one is valid.

1. \( \forall x. \exists y. P(x, y) \implies \exists y. \forall x. P(x, y) \)

2. \( \exists y. \forall x. P(x, y) \implies \forall x. \exists y. P(x, y) \)

**Solution.** The first implication, \( \forall x. \exists y. P(x, y) \implies \exists y. \forall x. P(x, y) \), is invalid.

One counter model is the predicate \( P(x, y) := y < x \) where the domain of discourse is the real numbers, \( \mathbb{R} \). For every real number \( x \), there exists a real number \( y \) which is strictly less than \( x \), so the antecedent of the implication is true. But there is no minimum real number, so the consequent is false.

The second implication is valid. Let’s say that “\( x \) is good for \( y \)” when \( P(x,y) \) is true. The hypothesis says that there is some element, call it \( g \), that is good for everything. The conclusion is that every element has something that is good for it, which of course is true since \( g \) will be good for it.

**STAFF NOTE:** It’s not clear students will be able to articulate the validity explanation. If they get stuck, offer them the “\( x \) is good for \( y \)” phrase as helpful. If it doesn’t help, then explain the answer.

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**Problem 4.**

When the Poet says “There is a season for every purpose under heaven.” Which of the following does he mean:

1. \( \exists s \in \text{Season}. \forall p \in \text{Purpose}. s \text{ is for } p \) (2)
2. \( \forall p \in \text{Purpose}. \exists s \in \text{Season}. s \text{ is for } p \) (3)

**Briefly explain.**

**Solution.** This poetic statement is meant to offer solace: this may be a bad season for you now, but be hopeful, a season that suits your purpose will come. So the appropriate translation would be formula (3), namely that given your Purpose, you can find a season that’s good for it. For example, if your purpose is planting, take heart: even though it’s Winter now, Spring is coming.

Formula (2) says you can find a single season, say Spring, that’s good for every possible Purpose like skiing, leaf watching, . . . . This is false, so it’s clearly not what the Poet meant. But even though he really meant (3), he used his poetic license to express (3) in a way that mechanically would translate into (2).

Note that a similar statement, “There is a man for all seasons,” is famously used to describe one extraordinarily versatile man, Sir Thomas More. So this statement would actually best be translated as

\[ \exists x \in \text{men}. \forall s \in \text{seasons}. x \text{ is (good) for } s \]

**STAFF NOTE:** Students may not come to the conclusion above. That’s fine, as long as they have a lucid explanation of how they got to another conclusion.

Ask about the Thomas More sentence once they’ve settled (2) and (3).
Problem 5.
A certain cabal within the Math for Computer Science course staff is plotting to make the final exam *ridiculously hard.* (“Problem 1. Prove the Poincare Conjecture starting from the axioms of ZFC. Express your answer in khipu —the knot language of the Incas.”) The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of seven people:

{Shashank, Igor, Albert, Drew, Ali, David, Gabe}

The cabal is a subset of these seven. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate cabal indicates who is in the cabal; that is, $\text{cabal}(x)$ is true if and only if $x$ is a member. Translate each statement below into English and deduce who is in the cabal.

(a) $\exists x, y, z. (x \neq y \text{ AND } x \neq z \text{ AND } y \neq z \text{ AND } \text{cabal}(x) \text{ AND } \text{cabal}(y) \text{ AND } \text{cabal}(z))$

**Solution.** A direct English paraphrase would be “There exist people we’ll call $x, y,$ and $z$, who are all different, such that $x, y,$ and $z$ are each in the cabal.” A better version would use the fact that there’s no need in this case to give names to the people. Namely, a better paraphrase is “There are at least 3 different people in the cabal.” Perhaps a simpler way to say this is: “The cabal is of size at least 3.”

(b) NOT($\text{cabal}(\text{Gabe}) \text{ AND } \text{cabal}(\text{Drew}))$

**Solution.** Gabe and Drew are not both in the cabal. Equivalently: at least one of Gabe and Drew is not in the cabal.

(c) $\text{cabal}(\text{David}) \text{ IMPLIES } \forall x. \text{cabal}(x)$

**Solution.** If David is in the cabal, then everyone is.

(d) $\text{cabal}(\text{Drew}) \text{ IMPLIES } \text{cabal}(\text{Gabe})$

**Solution.** If Drew is in the cabal, then Gabe is also.

(e) $(\text{cabal}(\text{Ali}) \text{ OR } \text{cabal}(\text{Albert})) \text{ IMPLIES NOT(\text{cabal}(\text{Igor}))}$

**Solution.** If either of Ali or Albert is in the cabal, then Igor is not. Equivalently, if Igor *is* in the cabal, the neither Albert nor Ali is.

(f) $(\text{cabal}(\text{Ali}) \text{ OR } \text{cabal}(\text{Gabe})) \text{ IMPLIES NOT(\text{cabal}(\text{Shashank}))}$

**Solution.** If either of Ali or Gabe is in the cabal, then Shashank is not. Equivalently, if Shashank is *in* the cabal, the neither Ali nor Gabe is.

(g) Now use these facts to figure out exactly who is on the cabal.

**STAFF NOTE:** If a team is stuck, tell them that the cabal consists of exactly Gabe, Ali, and Albert and have them check that this set satisfies all the conditions. (See the end if the solution.) Then start them back on proving that this is the unique set that works.
Solution. So much for the translations. We now argue that the only cabal satisfying all six propositions above is one whose members are exactly Gabe, Ali, and Albert.

We first observe that by (b), there must be someone —either Gabe or Drew—who is not in the cabal. But if David were in the cabal, then by (c), everyone would be. So we conclude by contradiction that:

David is not in the cabal. \hspace{1cm} (4)

Next observe that if Drew was in the cabal, then by (d), Gabe would be too, contradicting (b). So by again contradiction, we conclude:

Drew is not in the cabal. \hspace{1cm} (5)

Now suppose Igor is in the cabal. Then by (e), Ali and Albert are not, and we already know David and Drew are not, so only three remain who could be in the cabal, namely, Igor, Gabe, and Shashank. But by (a) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

Lemma 5.1. If Igor is in the cabal, then Gabe and Shashank are in the cabal.

But by (f), if Gabe is the cabal, then Shashank is not. That is,

Lemma 5.2. Gabe and Shashank cannot both be in the cabal.

Now from Lemma 5.2 we conclude that the conclusion of Lemma 5.1 is false. So by contrapositive, the hypothesis of Lemma 5.1 must also be false, namely,

Igor is not in the cabal. \hspace{1cm} (6)

Finally, suppose Shashank is in the cabal. Then by (f), Ali and Gabe are not, and we already know David, Drew, and Igor are not. So the cabal must consist of at most two people (Albert and Shashank). This contradicts (a), and we conclude by contradiction that

Shashank is not in the cabal. \hspace{1cm} (7)

So the only remaining people who could be in the cabal are Albert, Ali, and Gabe. Since the cabal must have at least three members, we conclude that

Lemma 5.3. The only possible cabal consists of Albert, Ali, and Gabe.

But we’re not done yet: we haven’t shown that a cabal consisting of Albert, Ali, and Gabe actually does satisfy all six conditions. So let \( A = \{\text{Albert, Ali, Gabe}\} \), and let’s quickly check that \( A \) satisfies (a)–(f):

- \(|A| = 3\), so \( A \) satisfies (a).
- Drew is not in \( A \), so \( A \) satisfies (b) and (d).
- David is not in \( A \), so the hypothesis of (c) is false, which means that \( A \) satisfies (c).
- Finally, Igor and Shashank are not in \( A \), so the conclusions of both (e) and (f) are true, and so \( A \) satisfies (e) and (f).

So now we have proved

Proposition. \( \{\text{Albert, Ali, Gabe}\} \) is the unique cabal satisfying conditions (a)–(f).