Staff Solutions to In-Class Problems Week 1, Fri.

Problem 1.
Prove that if \( a \cdot b = n \), then \( a \) or \( b \) must be \( \leq \sqrt{n} \), where \( a, b, \) and \( n \) are nonnegative integers. Hint: by contradiction, Section 1.8.

Solution. Proof. Suppose to the contrary that \( a > \sqrt{n} \) and \( b > \sqrt{n} \). Then
\[
a \cdot b > \sqrt{n} \cdot \sqrt{n} = n,
\]
contradicting the fact that \( a \cdot b = n \).

Problem 2.
Generalize the proof of Theorem 1.8.1 repeated below that \( \sqrt{2} \) is irrational. For example, how about \( \sqrt{3} \) ? Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Theorem. \( \sqrt{2} \) is an irrational number.

Proof. The proof is by contradiction: assume that \( \sqrt{2} \) is rational, that is,
\[
\sqrt{2} = \frac{n}{d},
\]
where \( n \) and \( d \) are integers. Now consider the smallest such positive integer denominator, \( d \).

We will prove in a moment that the numerator, \( n \), and the denominator, \( d \), are both even. This implies that
\[
\frac{n}{2} \quad \text{and} \quad \frac{d}{2}
\]
is a fraction equal to \( \sqrt{2} \) with a smaller positive integer denominator, a contradiction.

Since the assumption that \( \sqrt{2} \) is rational leads to this contradiction, the assumption must be false. That is, \( \sqrt{2} \) is indeed irrational. This italicized comment on the implication of the contradiction normally goes without saying, but since this is an early example of proof by contradiction, we’ve said it.

To prove that \( n \) and \( d \) have 2 as a common factor, we start by squaring both sides of (1) and get
\[
2 = \frac{n^2}{d^2}, \quad \text{so} \quad 2d^2 = n^2. \quad (2)
\]
So 2 is a factor of \( n^2 \), which is only possible if 2 is in fact a factor of \( n \).

This means that \( n = 2k \) for some integer, \( k \), so
\[
n^2 = (2k)^2 = 4k^2. \quad (3)
\]
Combining (2) and (3) gives \( 2d^2 = 4k^2 \), so

\[
d^2 = 2k^2. \tag{4}
\]

So 2 is a factor of \( d^2 \), which again is only possible if 2 is in fact also a factor of \( d \), as claimed.

**Solution.** Proof. We prove that for any \( n > 1 \), \( \sqrt[n]{2} \) is irrational by contradiction.

Assume that \( \sqrt[n]{2} \) is rational. Under this assumption, there exist integers \( a \) and \( b \) with \( \sqrt[n]{2} = a/b \), where \( b \) is the smallest such positive integer denominator. Now we prove that \( a \) and \( b \) are both even, so that

\[
a/2 \\
b/2
\]

is a fraction equal to \( \sqrt[n]{2} \) with a smaller positive integer denominator, a contradiction.

\[
\sqrt[n]{2} = \frac{a}{b} \\
2 = \frac{a^n}{b^n} \\
2b^n = a^n.
\]

The lefthand side of the last equation is even, so \( a^n \) is even. This implies that \( a \) is even as well (see below for justification).

In particular, \( a = 2c \) for some integer \( c \). Thus,

\[
2b^n = (2c)^n = 2^n c^n, \\
b^n = 2^{n-1} c^n.
\]

Since \( n - 1 > 0 \), the righthand side of the last equation is an even number, so \( b^n \) is even. But this implies that \( b \) must be even as well, contradicting the fact that \( a/b \) is in lowest terms.

Now we justify the claim that if \( a^n \) is even, so is \( a \).

There is a simple proof by contradiction: suppose to the contrary that \( a \) is odd. It’s a familiar (and easily verified\(^1\)) fact that the product of two odd numbers is odd, from which it follows that the product of any finite number of odd numbers is odd, so \( a^n \) would also be odd, contradicting the fact that \( a^n \) is even.

More generally for any integers \( m, k > 0 \), if \( m^k \) is divisible by a prime number, \( p \), then \( m \) must be divisible by \( p \). This follows from the unique factorization of an integer into primes (see Section 8.3): the primes in the factorization of \( m^k \) are precisely the primes in the factorization of \( m \) repeated \( k \) times.

**STAFF NOTE:** A similar but somewhat more interesting generalization to suggest to students who finish quickly is \( \sqrt[3]{3} \) is irrational for \( n > 1 \). The proof is the same as above with “2” replaced by “3,” except that now the needed claim is that if \( a^n \) is divisible by 3, then so is \( a \), which requires appealing to prime factorization as in the previous paragraph.

**Problem 3.**
If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering \( \sqrt[2]{\sqrt[2]{2}} \) and arguing by cases.

\(^1\)Two odd integers can be written as \( 2x + 1 \) and \( 2y + 1 \) for some integers \( x \) and \( y \). Then their product is also odd because it equals \( 2z + 1 \) where \( z = 2(2xy + x + y) + 1 \).
Solution. We want to find irrational numbers $a, b$ such that $a^b$ is rational. We argue by cases.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Let $a = b = \sqrt{2}$. $a$ and $b$ are irrational since $\sqrt{2}$ is irrational as we know. Also, $a^b$ is rational by case hypothesis. So we have found the required $a$ and $b$ in this case.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then $a$ is irrational by case hypothesis, we know $b$ is irrational, and

$$a^b = \left(\sqrt{2}\right)^{\sqrt{2}} = \sqrt{2}\cdot\sqrt{2} = \sqrt{2}^2 = 2,$$

which is rational. So we have found the required $a$ and $b$ in this case also.

So in any case, there will be irrational $a, b$ such that $a^b$ is rational. Note that we have no clue about which case is true, but that didn’t matter.

Problem 4.

Here is a different proof that $\sqrt{2}$ is irrational, taken from the American Mathematical Monthly, v.116, #1, Jan. 2009, p.69:

Proof. Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer, $q > 0$, such that $\left(\sqrt{2} - 1\right) q$ is a nonnegative integer. Let $q' := \left(\sqrt{2} - 1\right) q$. Clearly $0 < q' < q$. But an easy computation shows that $\left(\sqrt{2} - 1\right) q'$ is a nonnegative integer, contradicting the minimality of $q$.

(a) This proof was written for an audience of college teachers, and at this point it is a little more concise than desirable. Write out a more complete version which includes an explanation of each step.

Solution. The points that need justification are:

1. Why is there a positive integer, $q$, such that $\left(\sqrt{2} - 1\right) q$ is a nonnegative integer? Answer: Since $\sqrt{2}$ is rational, so is $\sqrt{2} - 1$. So $\sqrt{2} - 1$ can be expressed as an integer quotient with positive denominator; now just let $q$ be that denominator.

2. Why is there such a least positive integer, $q$? Answer: As long as there is one such positive integer, there has to be a least one. This obvious fact is known as the Well Ordering Principle.

3. Why is $0 < q' < q$? Answer: We know that $1 < \sqrt{2} < 2$, so $0 < \sqrt{2} - 1 < 1$. Therefore, $0 < \left(\sqrt{2} - 1\right) r < r$ for any real number $r > 0$.

4. Why is $\left(\sqrt{2} - 1\right) q'$ a nonnegative integer? Answer: It’s actually positive, because it is a product of positive numbers. It’s integer because

$$\left(\sqrt{2} - 1\right) q' = \left(\sqrt{2} - 1\right)^2 q = 2q - 2q \sqrt{2} + q = q - 2 \cdot \left[\left(\sqrt{2} - 1\right) q\right],$$

and the last term is of the form $\text{(integer)} - 2 \cdot \text{[integer]}$.

(b) Now that you have justified the steps in this proof, do you have a preference for one of these proofs over the other? Why? Discuss these questions with your teammates for a few minutes and summarize your team’s answers on your whiteboard.
Solution. Both proofs seem about equally easy to understand. The previous problems shows that the first proof generalizes pretty directly from square roots to $k$th roots, which doesn’t seems as clear for the this second proof. On the other hand, the first proof requires appeal to Unique Prime Factorization, while the second just uses simple algebra.