Problem Set 8 Solutions

Due: Tuesday, November 2 @ 7pm

Problem 1. [25 points] Find $\Theta$ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1) + n$ + 1/n + n

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Solution. We use the method of Akra-Bazzi for these problems.

(a) We see that $a = 8, b = 1/2, h = \lfloor n/2 \rfloor - n/2$ so $p = 3$ gives $ab^p = 1$.

$$T(n) = \Theta(n^3(1 + \int_1^n u^4 du)) = \Theta(n^3(1 + \int_1^n u^{-3} du)) = \Theta(n^3).$$

(b) $a_1 = 2, b_1 = 1/8, h_1(n) = \lfloor n/8 \rfloor - n/8 + 1/n, g(n) = n, p = 1/3,$

$$T(n) = \Theta\left(n^{1/3} \left(1 + \int_1^n u^{1/3} du\right)\right)$$

$$= \Theta\left(n^{1/3} \left(1 + \int_1^n u^{4/3} du\right)\right)$$

$$= \Theta\left(n^{1/3} + n^{1/3} \int_1^n u^{-1/3} du\right)$$

$$= \Theta(n^{1/3} + n^{1/3} \frac{3}{2} (n^{2/3} - 1))$$

$$= \Theta(n).$$
Problem 2. [30 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

\[ T(1) = 1 \text{ and } T(n) = 4T(n/2) + n \]

Then \( T(n) = O(n) \).

False Proof We show this by induction. Let \( P(n) \) be the proposition that \( T(n) = O(n) \).

Base Case: \( P(1) \) is true because \( T(1) = 1 = O(1) \).

Inductive Case: For \( n \geq 1 \), assume that \( P(n-1), \ldots, P(1) \) are true. We then have that

\[ T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n) \]

And we are done.

(a) [5 pts] Identify the flaw in the above proof.
(b) [10 pts] A simple attempt to prove \( T(n) \neq O(n) \) via induction ultimately fails. We assume for sake of contradiction that \( T(n) = O(n) \). Then there exists positive integer \( n_0 \) and positive real number \( c \) such that for all \( n \geq n_0 \), \( T(n) \leq cn \). We then define \( P(n) \) as the proposition that \( T(n) \leq cn \).

We then proceed with strong induction.

**Base Case**, \( n = n_0 \): \( P(n_0) \) is true, by assumption.

**Inductive Step**: We assume \( P(n_0), P(n_0 + 1), \ldots, P(n - 1) \) true.

Fill in the rest of this proof attempt, and explain why it doesn’t work.

*Note: As this problem was updated so late, the graders will be instructed to be exceedingly lenient when grading this.*

(c) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(d) [10 pts] We have now seen several recurrences of the form \( T(n) = aT([n/b]) + n \). Some of them give a runtime that is \( O(n) \), and some don’t. Find the relationship between \( a \) and \( b \) that yields \( T(n) = O(n) \), and prove that this is sufficient.

**Solution.** (a) The flaw is that \( P(n) \) is a predicate on \( n \), whereas \( O(n) \) is a statement not on \( n \), but on the limit of \( n \) as \( n \) approaches infinity. \( T(n) = O(n) \) does not depend on the value of \( n \) - it is either true or false.

(b) We first take some \( n \geq 2n_0 \). Then,

\[
T(n) = 4T(n/2) + n
\]

From the inductive hypothesis, \( n/2 \geq n_0 \), so \( T(n/2) \leq cn/2 \). So this means that

\[
T(n) \leq 4cn/2 + n = 2cn + n = n(2c + 1)
\]

Which is not less than \( cn \). So the induction is simply not powerful enough.

(c) We have that \( p = 2 \), so \( T = \Theta(n^2(1 + \int_1^n (u/u^3)du)) = \Theta(n^2) \).

(d) From analyzing the integral we can see that any case where \( p < 1 \) will give a linear solution, so having the condition \( a < b \) is sufficient.

Problem 3. [15 points] Define the sequence of numbers \( A_i \) by

\[
A_0 = 2
\]

\[
A_{n+1} = A_n/2 + 1/A_n \quad \text{(for } n \geq 1)\]

Prove that \( A_n \leq \sqrt{2} + 1/2^n \) for all \( n \geq 0 \).
Solution. Proof. The proof is by induction on \( n \). Let \( P(n) \) be the proposition that \( A_n \leq \sqrt{2} + 1/2^n \).

Base case: \( A_0 = 2 \leq \sqrt{2} + 1/2^0 \) is true.

Inductive step: Let \( n \geq 0 \) and assume the inductive hypothesis \( A_n \leq \sqrt{2} + 1/2^n \). We need the following lemma.

Lemma. For real numbers \( x > 0 \), \( x/2 + 1/x \geq \sqrt{2} \).

Proof. For real numbers \( x > 0 \),
\[
\frac{x}{2} + \frac{1}{x} \geq \sqrt{2} \\
\iff x^2 + 2 \geq 2\sqrt{2} \cdot x \\
\iff x^2 - 2\sqrt{2} \cdot x + 2 \geq 0 \\
\iff (x - \sqrt{2})^2 \geq 0,
\]
which is true.

By using induction it is straightforward to prove that \( A_n > 0 \) for \( n \geq 0 \) (base case: \( A_0 = 2 > 0 \); inductive step: if \( A_n > 0 \), then \( A_{n+1} = A_n/2 + 1/A_n > 0 \)). By the lemma, \( A_n \geq \sqrt{2} \) for \( n \geq 0 \). Together with the inductive hypothesis this can be used in the following derivation:
\[
A_{n+1} = A_n/2 + 1/A_n \\
\leq (\sqrt{2} + 1/2^n)/2 + 1/\sqrt{2} \\
= \sqrt{2} + 1/2^{n+1}.
\]

This completes the proof.

Problem 4. [30 points] Find closed-form solutions to the following linear recurrences.

(a) [15 pts] \( x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \) \( (x_0 = 3, x_1 = 4, x_2 = 14) \)

Solution. The characteristic equation is \( r^3 - 4r^2 + r + 6 = 0 \).

Generally, solving a cubic equation is a difficult problem. However, we can find from inspection that the roots are:
\[
r_1 = -1 \\
r_2 = 2 \\
r_3 = 3
\]

Therefore a general form for a solution is
\[
x_n = A(-1)^n + B(2)^n + C(3)^n.
\]
Substituting the initial conditions into this general form gives a system of linear equations.

\[
\begin{align*}
3 &= A + B + C \\
4 &= -A + 2B + 3C \\
14 &= A + 4B + 9C
\end{align*}
\]

The solution to this linear system is \( A = 1 \), \( B = 1 \), and \( C = 1 \). The complete solution to the recurrence is therefore

\[ x_n = (-1)^n + 2^n + 3^n. \]

\[ \blacksquare \]

**b) [15 pts]** \( x_n = -x_{n-1} + 2x_{n-2} + n \) \( (x_0 = 5, x_1 = -4/9) \)

**Solution.** First, we find the general solution to the homogenous recurrence. The characteristic equation is \( r^2 + r - 2 = 0 \). The roots of this equation are \( r_1 = 1 \) and \( r_2 = -2 \). Therefore, the general solution to the homogenous recurrence is

\[ x_n = A(-1)^n + B2^n. \]

Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess there is a linear solution, that is, a solution of the form \( an + b \). If the solution is of this form, we must have

\[ an + b = -a(n - 1) - b + 2a(n - 2) + 2b + n \]

Gathering up like terms, this simplifies to

\[ n(a + a - 2a - 1) + (b + a + b + 4a - 2b) = 0 \]

which implies that

\[ n = -5a \]

But \( a \) is a constant, so this cannot be so. So we make another guess, this time that there is a quadratic solution of the form \( an^2 + bn + c \). If the solution is of this form, we must have

\[ an^2 + bn + c = -[a(n - 1)^2 + b(n - 1) + c] + 2[a(n - 2)^2 + b(n - 2) + c] + n \]

which simplifies to

\[ n^2(a + a - 2a) + n(b + b - 2a + 8a - 2b - 1) + (c + a - b + c - 8a + 4b - 2c) = 0 \]

This simplifies to

\[ n(6a - 1) + (-7a + 3b) = 0 \]
This last equation is satisfied only if the coefficient of \( n \) and the constant term are both zero, which implies \( a = 1/6 \) and \( b = 7/18 \). Apparently, any value of \( c \) gives a valid particular solution. For simplicity, we choose \( c = 0 \) and obtain the particular solution:

\[
x_n = \frac{1}{6} n^2 - \frac{7}{18} n.
\]

The complete solution to the recurrence is the homogenous solution plus the particular solution:

\[
x_n = A(-1)^n + B 2^n + \frac{1}{6} n^2 - \frac{7}{18} n
\]

Substituting the initial conditions gives a system of linear equations:

\[
\begin{align*}
5 &= A + B \\
-4/9 &= -A + 2B - +1/6 + 7/18
\end{align*}
\]

The solution to this linear system is \( A = 3 \) and \( B = 2 \). Therefore, the complete solution to the recurrence is

\[
x_n = 3 + 2(-2)^n + \frac{1}{6} n^2 + \frac{7}{18} n
\]