Notes for Recitation 8

1 The Grow Algorithm

Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted $N$-vertex graph $G$.

ALG-GROW:

1. Label the edges of the graph $e_1, e_2, \ldots, e_t$ so that $wt(e_1) \leq wt(e_2) \leq \ldots \leq wt(e_t)$.
2. Let $S$ be the empty set.
3. For $i = 1 \ldots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend $S$ with the edge $e_i$.
4. Output $S$.

In summary, ALG-GROW selects edges one at a time, always choosing the minimum weight edge that does not create a cycle with previously selected edges. Notice that as edges are added $S$ may not be connected. When the algorithm terminates, $S$ contains $N – 1$ edges. If it is connected, then it is a spanning tree.

Consider, for example, the following edge-weighted graph.

Now suppose we run ALG-GROW on our graph. We may choose the weight 1 edge on the bottom of the triangle of weight 1 edges in our graph. In the next step, we may choose the weight 1 edge on the top of the graph. Note that this edge still has minimum weight, and does not cause us to form a cycle, so ALG-GROW can choose it. We will then choose
one of the remaining weight 1 edges. Note that neither causes us to form a cycle. Continuing
the algorithm, we may end up with the same spanning tree shown below.

In this recitation, we will analyze ALG-GROW.

2 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

Lemma 1. Suppose that \( T = (V, E) \) is a simple, connected graph. Then \( T \) is a tree iff
\(|E| = |V| - 1|.

In this exercise you will prove the following theorem.

Theorem. For any connected, weighted graph \( G \), ALG-GROW produces an MST of \( G \).

(a) Prove the following lemma.

Lemma 2. Let \( T = (V, E) \) be a tree and let \( e \) be an edge not in \( E \). Then, \( G = (V, E \cup \{e\}) \) contains a cycle.

(Hint: Suppose \( G \) does not contain a cycle. Is \( G \) a tree?)

Solution. Proof. (by contradiction) Suppose \( G \) does not contain a cycle. By the definition of a tree, \( T \) is connected. Notice that \( T \) is a subgraph of \( G \). Because any two
nodes in \( G \) are connected by a path in \( T \), \( G \) is a connected graph. So \( G \) is connected
and acyclic and therefore a tree by definition. Both \( G \) and \( T \) are trees and have the
same number of nodes. Therefore, they have the same number of edges (by Lemma 1).
This is a contradiction because \( G \) has one more edge than \( T \).

(b) Prove the following lemma.
Lemma 3. Let $T = (V, E)$ be a spanning tree of $G$ and let $e$ be an edge not in $E$. Then there exists an edge $e' \neq e$ in $E$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.

(Hint: Adding $e$ to $E$ introduces a cycle in $(V, E \cup \{e\})$.)

Solution. Proof. By Lemma 2, we know that the set of edges $E \cup \{e\}$ contains a cycle. If this cycle does not contain the edge $e$, then this cycle is a subset of $E$. Since $E$ is the set of edges of a tree, this cannot occur. So, this cycle contains $e$. If $e'$ is another edge distinct from $e$ in this cycle, then the graph $T^*$ that results after removing $e'$ from $E \cup \{e\}$ is still connected. The number of edges in $T^*$ is equal to the number of edges in $T$, which is equal to $|V| - 1$ by Lemma 1. Since $T^*$ is connected, $T^*$ is a tree by Lemma 1. Since $T^*$ is a subgraph of $G$ with vertices $V$, it spans $G$. $\square$

(c) Prove the following lemma.

Lemma 4. Let $T = (V, E)$ be a spanning tree of $G$, let $e$ be an edge not in $E$ and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct $T^*$, at least one is not in $S$.)

Solution. Proof. We need to change the proof in part (b) slightly. The proof of part (b) holds for any edge $e' \neq e$ in the cycle. We need to show that we can select an edge $e' \neq e$ that is in the cycle but not in $S$. We will prove this by contradiction. Suppose that all the edges not equal to $e$ that are in the cycle are in $S$. Then, $S \cup e$ is a cycle. This contradicts the assumption of the lemma. $\square$

(d) Prove the following lemma.

Lemma 5. Define $S_m$ to be the set consisting of the first $m$ edges selected by ALG-GROW from a connected graph $G$. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of $G$. Then $\forall m. P(m)$.

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

Solution. Proof. (By induction.) Let $P(m)$ be the predicate as defined above.

Base Case: $S_0$ contains 0 edges and is equal to the empty set, which is a subset of any set of edges $E$.

Inductive Step: Assume $P(m)$ in order to prove $P(m + 1)$.
If $m \geq |V|$ then $m + 1 > |V|$ and $P(m + 1)$ holds vacuously. Otherwise, if $m < |V|$ then let $e$ denote the $(m + 1)$th edge selected by ALG-GROW. By the inductive hypothesis, there exists an MST $T = (V, E)$ such that $S_m \subseteq E$. There are now two cases.

In the first case, $e \in E$ which case $S_m \cup \{e\} \subseteq E$, and thus $P(m + 1)$ holds.

In the second case, $e \notin E$, as illustrated by the following diagram. Now we need to find a different MST that contains $S_m \cup \{e\}$.

![Diagram of graph with edges and set notation]

What happens when we add $e$ to $T$? By the description of ALG-GROW, $S_m \cup \{e\}$ does not contain a cycle. Therefore, by Lemma 4, there exists an edge $e' \neq e$ in $E - S_m$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree for $G$.

In order to prove that $T^*$ is a MST, we need to show that $wt(e) \leq wt(e')$. We will prove this by contradiction. Suppose that $wt(e') < wt(e)$. Since $e' \in E$, which is the set of edges of the MST $T$, and $S_m \subseteq E$, the set of edges $S_m \cup \{e'\}$, does not contain a cycle. Therefore $e'$ would have already been added to $S_m$ in a previous iteration of ALG-GROW as one of the first $m$ edges. However, $e'$ is in $E - S_m$. This is a contradiction.

(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for $G$ such that $S \subseteq E$. Use contradiction to rule out the case in which $S$ is a proper subset of $E$.)

**Solution.** Proof. (by contradiction) Let $S$ be the set of edges produced by ALG-GROW. By Lemma 5, there exists an MST $T = (V, E)$ for $G$ such that $S \subseteq E$. If $S = E$, then ALG-GROW outputs the edges of the MST $T$.

We will show that the other case, $S \neq E$, leads to a contradiction. Suppose $S \neq E$. Then there exists an edge $e \in E - S$. This implies that $S \cup \{e\} \subseteq E$. Since $E$ is the set of edges of a tree, $S \cup \{e\}$ does not contain a cycle. Therefore, $e$ would be added to $S$ by ALG-GROW. So $e \in S$, and this contradicts $e \in E - S$. 

\[\square\]
Unique MST Extension

For a graph in which all edge weights are different, there exists a unique MST, which is produced by ALG-GROW. To prove this, modify the inductive hypothesis in Lemma 5 to address a unique MST, and show that uniqueness holds in the induction step due to the unique ordering of edges by weight.