Quiz 1

Problem 1. [10 points] In problem set 1 you showed that the **nand** operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator **universal**. Another universal operator is **nor**, defined such that \( P \text{nor} Q \equiv \neg(P \lor Q) \). Show how to express \( P \land Q \) in terms of: **nor**, \( P \), \( Q \), and grouping parentheses.

**Solution.** \((\neg P) \text{nor} (\neg Q) = (P \text{nor} P) \text{nor} (Q \text{nor} Q)\). ■

Problem 2. [15 points] We define the sequence of numbers

\[
a_n = \begin{cases} 
1 & \text{if } 0 \leq n \leq 3, \\
a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4.
\end{cases}
\]

Prove that \( a_n \equiv 1 \pmod{3} \) for all \( n \geq 0 \).

**Solution.** Proof by strong induction. Let \( P(n) \) be the predicate that \( a_n \equiv 1 \pmod{3} \).

Base case: For \( 0 \leq n \leq 3 \), \( a_n = 1 \) and is therefore \( \equiv 1 \pmod{3} \).

Inductive step: For \( n \geq 4 \), assume \( P(k) \) for \( 0 \leq k \leq n \) in order to prove \( P(n+1) \).

In particular, since each of \( a_{n-4}, a_{n-3}, a_{n-2} \) and \( a_{n-1} \) is \( \equiv 1 \pmod{3} \), their sum must be \( \equiv 4 \equiv 1 \pmod{3} \). Therefore, \( a_n \equiv 1 \pmod{3} \) and \( P(n+1) \) holds. ■

Problem 3. [20 points] The Slipped Disc Puzzle consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of exactly four tiles. As shown below, there are three ways to manipulate the puzzle:

**Shift Right:** The center disc is moved one unit to the right (if there is space)

**Rotate Disc:** The four tiles in the center disc are reversed

**Shift Left:** The center disc is moved one unit to the left (if there is space)
Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

Write your proof on the next page...

**Solution.** Order the tiles from left to right in the puzzle. Define an *inversion* to be a pair of tiles that is out of their natural order (e.g. 4 appearing to the left of 3).

**Lemma.** *Starting from the initial state there is an odd number of inversions after any number of transitions.*

**Proof.** The proof is by induction. Let \( P(n) \) be the proposition that starting from the initial state there is an odd number of inversions after \( n \) transitions.

**Base case:** After 0 transitions, there is one inversion, so \( P(0) \) holds.

**Inductive step:** Assume \( P(n) \) is true. Say we have a configuration that is reachable after \( n + 1 \) transitions.

1. Case 1: The last transition was a shift left or shift right

   In this case, the left-to-right order of the discs does not change and thus the number of inversions remains the same as in
2. The last transition was a rotate disc.

In this case, six pairs of disks switch order. If there were $x$ inversions among these pairs after $n$ transitions, there will be $6 - x$ inversions after the reversal. If $x$ is odd, $6 - x$ is odd, so after $n + 1$ transitions the number of inversions is odd.

Conclusion: Since all reachable states have an odd number of inversions and the goal state has an even number of inversions (specifically 0), the goal state cannot be reached.
Problem 4. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range \{0, 1, \ldots, 71\}.

Solution. Since 17 and 72 = 2^3 \cdot 3^2 are relatively prime, an inverse exists and can be found by either Euler’s theorem or the Pulverizer.

Solution 1: Euler’s Theorem

\[
\phi(72) = \phi(2^3 \cdot 3^2) \\
= \phi(2^3) \cdot \phi(3^2) \\
= (2^3 - 2^2)(3^2 - 3^1) \\
= 4 \cdot 6 = 24
\]

Therefore, \(17^{\phi(72)} = 17^{23}\) is an inverse of 17. To find the inverse in the range \{0, 1, \ldots, 71\} we take the remainder using the method of repeated squaring:

\[
\begin{align*}
17 &= 17 \\
17^2 &= 289 \\
\equiv 1 & \quad \text{(since 289 = 4 \cdot 72 + 1)} \\
17^4 &\equiv 1^2 = 1 \\
17^8 &\equiv 1 \\
\ldots &\text{ etc.}
\end{align*}
\]

Therefore the inverse of 17 in the range \{0, 1, \ldots, 71\} is given by,

\[
17^{23} = 17^{16}17^417^217^1 \\
\equiv 1 \cdot 1 \cdot 1 \cdot 17 \\
= 17
\]

Solution 2: The Pulverizer

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<td>17</td>
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<td>(17 - 4 \cdot 4)</td>
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<td>(= 17 - 4 \cdot (72 - 4 \cdot 17))</td>
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<td>4</td>
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Since \(17^2 - 4 \cdot 72 = 1\), \(17^2 \equiv 1 \pmod{72}\) and so 17 is self inverse. ■
Problem 5. [15 points] Consider a graph representing the main campus buildings at MIT.

(a) [5 pts] Give the diameter of this graph:

Solution. The diameter is 6, the length of a shortest path between buildings 1 and 2.

(b) [5 pts] Is this graph bipartite? Provide a brief argument for your answer.

Solution. No, there is an odd-length cycle.

(c) [5 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer.

Solution. This graph does not have an Euler circuit because there are vertices with odd degree.

Problem 6. [10 points]

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from $u$ to $v$ or an edge from $v$ to $u$ for every distinct pair of nodes $u$ and $v$. (The nodes represent players and an edge $u \rightarrow v$ indicates that player $u$ beats player $v$.)

Consider the “beats” relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold for all tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. transitive

Solution. The “beats” relation is not transitive because there could exist a cycle of length 3 where $x$ beats $y$, $y$ beats $z$ and $z$ beats $x$. By the definition of a tournament, $x$ cannot then beat $y$ in such a situation.

2. symmetric

Solution. The “beats” relation is not symmetric by the definition of a tournament: if $x$ beats $y$ then $y$ does not beat $x$. 

3. antisymmetric

Solution. The “beats” relation is antisymmetric since for any distinct players $x$ and $y$, if $x$ beats $y$ then $y$ does not beat $x$. ■

4. reflexive

Solution. The “beats” relation is not reflexive since a tournament graph has no self-loops. ■

Problem 7. [20 points] An outerplanar graph is an undirected graph for which the vertices can be placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, $K_4$, is not outerplanar but any proper subgraph of $K_4$ with strictly fewer edges is outerplanar. Some examples are provided below:

![Examples of outerplanar graphs](image)

Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Solution. Proof. Proof by induction on the number of nodes $n$ with the induction hypothesis $P(n) =$ ”every outerplanar graph with $n$ vertices is 3-colorable.”

Base case: For $n = 1$ the single node graph with no edges is trivially outerplanar and 3-colorable.

Inductive step: Assume $P(n)$ holds and let $G_{n+1}$ be an outerplanar graph with $n+1$ vertices. There must exist a vertex $v$ in $G_{n+1}$ with degree at most 2. Removing $v$ and all its incident edges leaves a subgraph $G_n$ with $n$ vertices.

Since $G_{n+1}$ could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so $G_n$ is also an outerplanar graph. $P(n)$ implies $G_n$ is 3-colorable. Therefore we can color all the vertices in $G_{n+1}$ other than $v$ using only 3 colors and since $\deg(v) \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. Therefore, $G_{n+1}$ is 3-colorable and $P(n+1)$ holds.

□