Problem Set 6 Solutions

Due: Monday, October 16 at 8pm

Problem 1. [10 points] In this problem we continue the study of planar graphs, as studied in recitation 10. Show that any planar graph can be colored with only 6 colors.

Solution. The proof is by induction on the number of vertices of a planar graph $G$. Our induction hypothesis is $P(n)$: any planar graph on $n$ vertices can be colored with only 6 colors. We show that $P(n)$ implies $P(n+1)$.

Base case ($n = 1$): There is one vertex and no edges, and so $G$ is trivially 6-colorable.

Inductive step: Now assume $P(n)$ for some $n \geq 1$. We show that $P(n+1)$ also holds. Let $G = (V, E)$ be a planar graph with $n+1$ vertices. By problem 2 in section 2 of recitation 10, we know that $G$ contains a vertex $v$ of degree at most 5. Consider the graph $G'$ obtained from $G$ by removing $v$ and its incident edges. Observe that $G'$ is also a planar graph, since if we take the planar embedding of $G$ and remove the curves associated with $v$ and its incident edges, we obtain a planar embedding of $G'$ (that is, we cannot cause two existing curves to intersect).

$G'$ is a planar graph with only $n$ vertices. By the inductive hypothesis $P(n)$, it can be colored with only 6 colors. Fix a coloring $\chi : V \setminus \{v\} \to \{1, 2, \ldots, 6\}$ of $G'$. We extend $\chi$ to a coloring of $G$. Since $v$ has degree at most 5, it is incident to at most 5 different colors. Therefore, there is at least one color $c$ in $\{1, 2, \ldots, 6\}$ not assigned to any of $v$'s neighbors. We choose to color $v$ with color $c$. It now follows that the coloring $\chi' : V \to \{1, 2, \ldots, 6\}$ defined by $\chi'(u) = \chi(u)$ if $u \neq v$, and $\chi'(v) = c$ is a coloring of $G$. Thus, $P(n+1)$ holds, and the proof is complete.

Problem 2. [15 points] In this problem we study some properties of relations. Recall that a relation $R \subseteq X \times Y$ is a set of pairs $(x, y)$.

(a) A function $F \subseteq X \times Y$ is a relation with the extra property that if $(x, y) \in F$ and $(x, y') \in F$, then $y = y'$. Let $Z = \{0, 1, 2, \ldots, p-1\}$. Consider the set $S$ of all pairs $(x, y) \in Z \times Z$ for which $x = y^2 \mod p$. Prove or disprove: $S$ is a function.

Solution. $S$ is not a function. To see this, observe that $(1, 1)$ and $(1, p-1)$ both occur in $S$, since $1 \equiv 1^2 \mod p$ and

$$1 \mod p \equiv (-1)^2 \mod p \equiv (p-1)^2 \mod p.$$
Recall that a special type of relation is an equivalence relation, that is, a relation that is reflexive, symmetric, and transitive. For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(b) \( R := \{(x, y) \in \{0, 1\}^n \times \{0, 1\}^n \mid \Delta(x, y) \leq 1\} \), where \( \Delta(x, y) \) denotes the Hamming distance of \( x \) and \( y \), that is, the number of coordinates which differ. For instance, the strings 000 and 010 have Hamming distance 1 since they differ on the second coordinate, and the strings 011 and 110 have Hamming distance 2 since they differ on the first and last coordinates.

**Solution.** \( R \) is not an equivalence relation. For \( b \in \{0, 1\} \), we use the notation \( b^i \) to denote \( i \) consecutive \( b \)s. Let \( x = 0^n, y = 10^{n-1}, \) and \( z = 120^{n-2} \). Then \( \Delta(x, y) = 1 \) and \( \Delta(y, z) = 1 \), but \( \Delta(x, y) = 2 \), so \( R \) is not transitive.

(c) \( R := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \exists n \in \mathbb{Z}, y = 2^n x\} \).

**Solution.** This is an equivalence relation. We need to show \( R \) is reflexive, symmetric, and transitive. First, \( x = 2^0 x, \) so \( (x, x) \in R \), and \( R \) is reflexive. Next, if \( (x, y) \in R \), then \( x = 2^n y \) for some \( n \in \mathbb{Z} \), which means that \( y = 2^{-n} x \), and so \( (y, x) \in R \), and \( R \) is symmetric. Finally, if \( (x, y) \in R \) and \( (y, z) \in R \), then \( x = 2^n y \) and \( y = 2^m z \) for \( n_1, n_2 \in \mathbb{Z} \), so \( x = 2^{n_1+n_2} z \), and thus \( (x, z) \in R \).

The equivalence classes are the infinite sets

\[
S_x = \{y \mid \exists n \in \mathbb{Z} \text{ such that } y = 2^n x\}.
\]

**Problem 3.** [15 points] In this problem we study partial orders (posets). Recall that a partial order \( \preceq \) on a set \( X \) is reflexive \( (x \preceq x) \), anti-symmetric \( (x \preceq y \land y \preceq x \implies x = y) \), and transitive \( (x \preceq y \land y \preceq z \implies x \preceq z) \). Note that it may be the case that neither \( x \preceq y \) nor \( y \preceq x \). A chain is a list of distinct elements \( x_1, \ldots, x_i \) in \( X \) for which \( x_1 \preceq x_2 \preceq \cdots \preceq x_i \). An antichain is a subset \( S \) of \( X \) such that for all distinct \( x, y \in S \), neither \( x \preceq y \) nor \( y \preceq x \).

The aim of this problem is to show that any sequence of \( (n-1)(m-1)+1 \) integers either contains a non-decreasing subsequence of length \( n \) or a decreasing subsequence of length \( m \). Note that the given sequence may be out of order, so, for instance, it may have the form 1, 5, 3, 2, 4 if \( n = m = 2 \). In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider 1, 2, 4 and 5, 3, 2).

(a) Label the given sequence of \( (n-1)(m-1)+1 \) integers \( a_1, a_2, \ldots, a_{(n-1)(m-1)+1} \). Show the following relation \( \preceq \) on \( \{1, 2, 3, \ldots, (n-1)(m-1)+1\} \) is a poset: \( i \preceq j \) if and only if \( i \leq j \) and \( a_i \leq a_j \) (as integers)

**Solution.** We show reflexivity, anti-symmetry, and transitivity. Clearly \( i \preceq i \) since \( i \leq i \) and \( a_i \leq a_i \), so \( \preceq \) is reflexive. Next, suppose \( i \preceq j \) and \( j \preceq i \). Then \( i \leq j \leq i \), so \( i = j \), and \( \preceq \) is anti-symmetric. Finally, suppose \( i \preceq j \) and \( j \preceq k \). Then \( i \leq j \) and \( j \leq k \), so \( i \leq k \). Moreover, \( a_i \leq a_j \) and \( a_j \leq a_k \), so \( a_i \leq a_k \). Thus, \( \preceq \) is transitive.
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For the next part, we will need to use Dilworth’s theorem, as covered in lecture. Recall that Dilworth’s theorem states that if \((X, \preceq)\) is any poset whose longest chain has length \(n\), then \(X\) can be partitioned into at most \(n\) disjoint antichains.

(b) Show that in any sequence of \((n - 1)(m - 1) + 1\) integers, either there is a non-decreasing subsequence of length \(n\) or a decreasing subsequence of length \(m\).

**Solution.** Consider the \(\preceq\) relation on \(\{1, 2, \ldots, (n - 1)(m - 1) + 1\}\) defined above. The length of the longest non-decreasing subsequence of the given integers is just the length of the longest chain in this poset. If the longest chain has length at least \(n\), we are done, so suppose the length of the longest chain is at most \(c \leq n - 1\).

Then, by the first part we know that \(\{1, 2, \ldots, (n - 1)(m - 1) + 1\}\) can be decomposed into \(c\) disjoint antichains. Consider the indices \(i_1 \leq i_2 \leq \cdots \leq i_s\) in any antichain \(A\). Then it must be the case that \(a_{i_1} > a_{i_2} > \cdots > a_{i_s}\), as otherwise we would have \(a_{i_j} \leq a_{i_{j'}}\) for some \(j < j'\), and thus \(j \preceq j'\), and \(A\) could not be an antichain. It follows that there is a decreasing subsequence of length at least \(|A|\).

Since we can partition \(\{1, 2, \ldots, (n - 1)(m - 1) + 1\}\) into at most \(c \leq n - 1\) disjoint antichains, one such antichain must have size at least

\[
\frac{(n - 1)(m - 1) + 1}{c} \geq \frac{(n - 1)(m - 1) + 1}{n - 1} \geq m - 1 + \frac{1}{n - 1} \geq m,
\]

which completes the proof.

(c) Construct a sequence of \((n - 1)(m - 1)\) integers, for arbitrary \(n\) and \(m\), that has no non-decreasing subsequence of length \(n\) and no decreasing subsequence of length \(m\). Thus in general, the result you obtained in the previous part is best-possible.

**Solution.** Consider the set of integers \(\{1, 2, \ldots, (n - 1)(m - 1)\}\). For each \(1 \leq i \leq n - 1\), define the decreasing subsequence of length \(m - 1\):

\[B_i = i(m - 1), \ldots, (i - 1)(m - 1) + 1.\]

Then the \(B_i\) partition \(\{1, 2, \ldots, (n - 1)(m - 1)\}\). Consider the sequence

\[S = B_1 \circ B_2 \circ \cdots \circ B_{n - 1}.\]

Any non-decreasing subsequence of \(S\) can contain at most one integer from any single \(B_i\), since the \(B_i\) are decreasing subsequences. Thus, the length of the longest non-decreasing subsequence is at most \(n - 1\).

Any decreasing subsequence must be entirely contained in a single \(B_i\), since for \(j > i\), any integer in \(B_j\) is larger than any integer in \(B_i\). Thus, the length of the longest decreasing subsequence is at most \(m - 1\).