Solutions to Problem Set 4

**Problem 1. (15 points)** Let $G = (V, E)$ be a graph. A matching in $G$ is a set $M \subseteq E$ such that no two edges in $M$ are incident on a common vertex.

Let $M_1$, $M_2$ be two matchings of $G$. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either $M_1$ or $M_2$). Show that $G'$ is bipartite.

We will need this result in one of the coming lectures.

**Solution.** We will show that $G'$ has no odd cycle. By the theorem proved in recitation (that a graph is bipartite iff it has no odd cycles) we will be done.

Take a sequence of edges $(e_1, e_2, \ldots, e_k)$ in $G'$ that form a cycle. Let us show that $k$ must be even. We know that $e_1 \in M_1 \cup M_2$. Assume $e_1 \in M_1$ (otherwise $e_1 \in M_2$ and the argument is identical replacing $M_1$ with $M_2$). Now since $e_1$ and $e_2$ are incident on a common vertex, and $M_1$ is a matching, $e_2$ cannot be in $M_1$. So $e_2 \in M_2$. Similarly $e_3 \in M_1$, $e_4 \in M_2$ etc. By induction we can prove that for all $i \leq k$, $e_i \in M_1$ iff $i$ is odd.

Now if $k$ had been odd, we would have $e_k \in M_1$. But $e_k$ and $e_1$ are adjacent edges in the cycle, and hence incident on a common vertex, we have a contradiction. Thus $k$ must be even.

**Problem 2.** Let $G = (V, E)$ be a graph. Recall that the degree of a vertex $v \in V$, denoted $d_v$, is the number of vertices $w$ such that there is an edge between $v$ and $w$.

- Prove that

\[2|E| = \sum_{v \in V} d_v.\]

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Solution. Let $S = \{(e, v) \in E \times V : e \text{ is incident on } v\}$.

Count the elements in $S$ as follows

$$|S| = \sum_{e \in E} |\{v : (e, v) \in S\}| = 2|E|$$

and also as

$$|S| = \sum_{v \in V} |\{e : (e, v) \in S\}| = \sum_{v \in V} d_v.$$

The result follows.

Once can also prove it by induction on $|E|$. You should try to.

At a 6.042 ice-cream study session (where ice-cream flows by the way, really, you should go ... and yeah, it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

Solution. Assume that the study is accurate. Define a graph $G = (V, E)$ with students as vertices and put an edge between 2 students if they shook hands. By the previous problem, we should have $2|E| = \sum_v d_v = 111 \cdot 17$. But the LHS is even and the RHS is odd, a contradiction.

And on a more dull note, how many edges does $K_n$, the complete graph on $n$ vertices, have?

Solution. Apply the first part of the problem. Notice that each vertex is joined to $n - 1$ others. $2|E| = \sum_v d_v = n(n - 1)$. So $|E| = n(n - 1)/2$.

Problem 3. Let $n$ be a positive integer. Consider the graph $G$ whose vertices are the elements of $\{1, 2, \ldots, 2n\}$, and whose edges are given by the following rule: there is an edge between vertex $i$ and $j$ iff $(i - j \equiv 1 \mod 2n) \lor (i - j \equiv -1 \mod 2n) \lor (i - j \equiv n \mod 2n))$.

For each $k \in \{1, 2, \ldots, 2n\}$, find the distance between vertex 1 and vertex $k$.

Solution. If $k \leq n/2$, the distance is $k - 1$. If $n/2 < k \leq n + 1$, the distance is $2n - (k + n) + 1$. If $n + 1 < k \leq 3n/2$, the distance is $k - n$. If $3n/2 < k \leq 2n$, the distance is $2n - k + 1$.
• What is the diameter of this graph?

**Solution.** $\lfloor n/2 \rfloor + 1$. ■

• Prove that this graph is not 4-edge-connected: that is, you can remove 3 edges and disconnect the graph.

**Solution.** Remove the 3 edges adjacent to vertex 1. ■

• Prove that this graph is 3-edge-connected: that is, if you remove two edges from the graph, the graph remains connected.

**Solution.** Suppose you could remove 2 edges and disconnect the graph. Now if only 1 edge is removed from the big cycle, the big cycle remains connected (and hence the graph too). Thus it must be that both the edges were removed from the big cycle. This breaks the big cycle into 2 components. However, there is still a diameter edge crossing from one component to the other, a contradiction. ■

• Describe the induced graph on the odd numbered vertices \{1, 3, \ldots, 2n - 1\}.

**Solution.** If $n$ is odd, then it is $n$ disconnected vertices. If $n$ is even it is a perfect matching on $n$ vertices. ■

• Describe the induced graph on the vertices \{1, 2, \ldots, n\}.

**Solution.** A cycle of length $n$. ■

• What is the chromatic number of $G$? (It may depend on $n$).

**Solution.** If $n$ is odd, it is 2 colorable (color vertex $i$ by color $\text{rem}(i, 2)$). If $n = 2$ it is 4 colorable. If $n$ is even and $\geq 4$, it is 3 colorable (color vertex 1, \ldots, $n$ with colors 1, 2, 1, 2, \ldots, 1, 2 and color vertices $n + 1, \ldots, 2n - 2$ with colors 3, 1, 3, 1, \ldots, 3, 1 and color vertices $2n - 1$ and $2n$ with colors 2, 3. It can be checked that this is a coloring). ■

**Problem 4.** Give an example of an instance of the stable marriage problem (i.e., a collection of boys and girls along with their preferences for each other) that has at least two stable matchings. What answer does the mating ritual give for your example? If you flip the roles of boys and girls in the mating ritual, what answer will you get?
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Solution. Take for example the stable matching instance given in the lecture notes. We ran the mating algorithm on this instance and arrived at one stable matching. On the other hand, clever little Jose sitting in class noticed that all the girls had different first preferences, so pairing them up with their first preference would give another stable matching. These matchings are distinct and correspond to the answer given by the mating algorithm and the flipped mating algorithm respectively. One can also find smaller examples where there are 2 stable matchings.

Problem 5. A numbered graph is a pair \((G, f)\) where \(G\) is a graph and \(f\) is a function \(f : V(G) \to \mathbb{R}\). More colloquially, it is a graph with a a number associated to every vertex. Numbered graph \((G, f)\) is said to be smooth if for each vertex \(v\), \(f(v)\) is the median of the values of \(f\) on the neighbors of \(v\) (choosing the lower one in case the median is not unique).

Claim: In any smooth numbered graph \((G, f)\) where \(G\) is connected, \(f\) must be a constant.

- Give a wrong proof of this by induction on the number of vertices of the graph using build-up error.
- What is the flaw in this proof?
- Find a counterexample to the claim.

Solution. Take 6 vertices \(a, b, c, d, e, f\), with \(a, b, c\) forming a triangle, \(d, e, f\) forming a triangle, and \(a\) connected to \(d\). Let \(a, b, c\) be given the number 1 and \(d, e, f\) given the number 2. It is easy to see that this is a smooth numbered connected graph.

- Give a (correct!) proof of the claim when the definition of smooth uses ”mean” instead of ”median”. Hint: look at the vertex \(v\) for which \(f(v)\) is minimum.

Solution. Let \((G, f)\) be a smooth connected numbered graph. Let \(x = \min_{v \in V} f(v)\). Let \(S = \{v \in V : f(v) = x\}\).

Claim: If \(v \in S\), and \(w\) is a neighbor of \(v\), then \(w \in S\).

Proof: Let \(N(v)\) be the set of neighbors of \(v\). By smoothness,

\[
|N(v)| f(v) = \sum_{v' \in N(v)} f(v').
\]

(1)

We know that for any \(v' \in N(v)\), \(f(v') \geq f(v)\). In particular \(f(w) \geq f(v)\). Suppose, for the sake of contradiction, that \(f(w) > f(v)\). Then

\[
\sum_{v' \in N(v)} f(v') = f(w) + \sum_{v' \in N(v), v' \neq w} f(v') > f(v) + (|N(v)| - 1) f(v) \geq |N(v)| f(v),
\]

which contradicts smoothness. Therefore, \(f(w) = f(v)\), and \(w \in S\).
a contradiction to (1). Thus the claim is true.

Let us resume the solution to the problem. There is at least 1 vertex in S. But we know that G is connected.

Lemma: If a $X$ is a nonempty set of vertices in a connected graph $G$ such that if $v \in X$, then any neighbor of $v$ is also in $X$, then $X$ is all of $V$.

Proof is by induction on the distance of a given vertex from $v$.

So the claim implies that all the vertices of $G$ are in $S$ and $f(v) = x$ for all $v$. $\blacksquare$