Problem Set 10

Due: Monday November 13, at 8 PM

Problem 1. [10 points] Ten women and thirteen men are on the faculty of a school’s math department. How many ways are there to select a committee of 7 members if

(a) at least 2 women must be on the committee?
(b) at least 1 man and 1 woman must be on the committee?
(c) there must be two more men than women on the committee?

Problem 2. [10 points] An \( n \)-bit Boolean function maps each \( n \)-bit string to a single bit; that is, an \( n \)-bit Boolean function is a function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \).

(a) Write a simple closed form for the number, \( b(n) \), of \( n \)-bit Boolean functions.

(b) The \( n \)-bit Boolean function, \( f \), depends on bit \( i \) iff there exist \( n \)-bit strings \( x \) and \( y \) such that \( x \) and \( y \) differ only in their \( i \)th bits and \( f(x) \neq f(y) \). Let \( c(n) \) be the number of \( n \)-bit Boolean functions that do not depend on their first bit. Write a simple closed form for \( c(n) \).

(c) Show that the number of \( n \)-bit Boolean functions \( a(n) \) that depend on all of their bits satisfies \( a(n) \sim b(n) \). Thus, “almost all” Boolean functions depend on all of their bits. Hint: It may be helpful to note that \( |\cup_i A_i| \leq \sum_i |A_i| \) for any collection of sets \( A_i \).

Problem 3. [10 points] You want to choose a team of \( m \) people from a pool of \( n \) people for your startup company, and from these \( m \) people you want to choose \( k \) to be the team managers. You took 6.042, so you know you can do this in
\[
\binom{n}{m} \binom{m}{k}
\]
ways. But your CFO, who went to Harvard Business School, comes up with the formula
\[
\binom{n}{k} \binom{n-k}{m-k}.
\]
Before you dump on your CFO or Harvard Business School, you decide to check his answer against yours — fortunately.
(a) Give an algebraic proof that your CFO’s formula agrees with yours.

(b) Now give what the Notes call a **combinatorial argument** proving this same fact.

**Problem 4.** [15 points] Suppose a generalized World Series (GSeries) between the Tigers and the Cardinals involved \(2n + 1\) games. As usual, the GSeries will stop as soon as one team has won \(n + 1\) games.

(a) Suppose that when the Cardinals finally win the GSeries, the Tigers have managed to win exactly \(r\) games (so \(r \leq n\)). How many possible win-loss patterns are possible for the Cardinals to win the GSeries in this way? Express your answer as a binomial coefficient, i.e., in the form \(\binom{m}{k}\) for certain integers \(m\) and \(k\).

(b) How many possible win-loss patterns are possible for the Cardinals to win the GSeries when the Tigers win at most \(r\) games? Express your answer as a binomial coefficient.

(c) Give a combinatorial proof that

\[
\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}.
\]

(d) Verify equation (4) by induction using algebra.

**Problem 5.** [10 points] In the following problem, you can leave your answers in closed form, that is, you do not have to write down the coefficients of the generating functions explicitly.

(a) Give a generating function for the number of ways to compose a bag of chocolate donuts, which contains at least 4 donuts.

(b) Give a generating function for the number of ways to compose a bag of glazed donuts, which contains at most 4 donuts.

(c) Give a generating function for the number of ways to compose a bag of coconut donuts, which contains exactly 1 donut.

(d) Give a generating function for the number of ways to compose a bag of plain donuts, where the number of donuts in the bag is a multiple of 9.

(e) Give a generating function for the number of ways to compose a bag of donuts where the donuts are of 4 kinds (chocolate, glazed, coconut and plain) and all the following conditions hold:

- There are at least 3 are chocolate donuts, and
Problem Set 10

- at most 4 donuts are glazed. Also,
- there are 2 coconut donuts, and
- the number of plain donuts is a multiple of 4.

Problem 6. [10 points] The \textbf{(ordinary) generating function} for a sequence \(\langle a_0, a_1, a_2, a_3, \ldots \rangle\) is the power series:

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) \(\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle\)

(b) \(\langle 1, 1, 1, 1, 1, 1, \ldots \rangle\)

(c) \(\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle\)

(d) \(\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle\)

(e) \(\langle 0, 0, 1, 1, 1, 1, 1, \ldots \rangle\)

(f) \(\langle 1, 3, 5, 7, 9, 11, \ldots \rangle\)

Problem 7. [10 points] Suppose that:

\[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \]
\[ g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \]

What sequences do the following functions generate?

(a) \(f(x) + g(x)\)
Problem 8. [10 points] Consider the following recurrence equation:

\[
T_n = \begin{cases} 
1 & n = 0 \\
2 & n = 1 \\
2T_{n-1} + 3T_{n-2} & (n \geq 2) 
\end{cases}
\]

Let \( f(x) \) be a generating function for the sequence \( \langle T_0, T_1, T_2, T_3, \ldots \rangle \).

(a) Give a generating function in terms of \( f(x) \) for the sequence:

\( \langle 1, 2, 2T_1 + 3T_0, 2T_2 + 3T_1, 2T_3 + 3T_2, \ldots \rangle \)

(b) Form an equation in \( f(x) \) and solve to obtain a closed-form generating function for \( f(x) \).

(c) Expand the closed form for \( f(x) \) using partial fractions.
(d) Find a closed-form expression for $T_n$ from the partial fractions expansion.

**Problem 9.** [15 points] A string of brackets is called “balanced” if the number of left brackets, $[,$ equals the number of right brackets, $]$ and the number of $]$’s never exceeds the number of $[$’s as the string is read from left to right. For example,

$$[[[[]]]]$$

is a balanced string, but $[ ] [ ]$ is not. There five different balanced strings with three $]$’s:

$$[[[ ][ ]]][ ][ ][ ][][ ][ ]$$

Your diligent TA stayed up until dawn, slaving away with a dull pencil stub, working out the following table:

<table>
<thead>
<tr>
<th># $]$’s</th>
<th># of Balanced Strings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
</tr>
<tr>
<td>8</td>
<td>1430</td>
</tr>
<tr>
<td>9</td>
<td>4862</td>
</tr>
<tr>
<td>10</td>
<td>16796</td>
</tr>
<tr>
<td>11</td>
<td>58786</td>
</tr>
<tr>
<td>12</td>
<td>208012</td>
</tr>
</tbody>
</table>

In an apparently unrelated calculation, your exhausted TA then used his final drops of energy and the last nubbin of his pencil to evaluate $500000 - 1000 \sqrt{249999}$ out to 72 decimal places:

$$1.000001000002000005000014000042000132000429001430004862016796058786208012 \ldots$$

Compare the table to the irrational number. The goal of this problem is to explain the apparently miraculous correspondence.

---

1. This is a lie; he cranked it out with a C program in about six minutes.
2. This is also a lie; he plugged the thing into Maple.
(a) Let $p_n$ be the number of balanced strings containing $n$ ‘s. Explain why the following recurrence holds:

$$
p_0 = 1, \quad \text{the empty string}
$$

$$
p_n = \sum_{k=1}^{n} p_{k-1} \cdot p_{n-k}, \quad \text{for } n \geq 1.
$$

(b) Let 

$$
P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots.
$$

be the generating function for the number of balanced strings. Prove that 

$$
P(x) = xP(x)^2 + 1.
$$

(c) Find a closed-form expression for the generating function $P(x)$.

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

(e) Explain why the digits of this irrational number encode these successive numbers of balanced strings.