Problem Set 9

Due: Monday November 8 at 9 PM

Problem 1. [5 points] Whenever two people at a party are introduced, they shake hands with each other. Explain why there must be two people at the party who shake the same number of hands. Hint: Pigeonhole.

Problem 2. [15 points] Seven women and nine men are on the faculty of a school’s math department. How many ways are there to select a committee of 5 members if

(a) at least 1 woman must be on the committee?

(b) at least 1 man and 1 woman must be on the committee?

(c) there must be more men than women on the committee?

Problem 3. [5 points] A pizza house is having a promotional sale. Their commercial reads:

We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can order each one with any combination of toppings absolutely free. That’s $22,369,621$ different ways to design your order!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3 / 3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3 / 3!$ is obviously not an integer, so clearly something is wrong.

Write a short, clear explanation so the ad writer can understand why his number is too small.

In how many ways can toppings for three pizzas really be chosen?

Problem 4. [10 points] An $n$-bit Boolean function maps each $n$-bit string to a single bit; that is, an $n$-bit Boolean function is a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$.

(a) Write a simple closed form for the number, $b(n)$, of $n$-bit Boolean functions.
(b) The $n$-bit Boolean function, $f$, depends on bit $i$ iff there exist $n$-bit strings $x$ and $y$ such that $x$ and $y$ differ only in their $i$th bits and $f(x) \neq f(y)$. Let $c(n)$ be the number of $n$-bit Boolean functions that do not depend on their first bit. Write a simple closed form for $c(n)$.

(c) We can say informally that “almost all” Boolean functions depend on all their bits. Give a precise mathematical interpretation of this informal statement (something about the asymptotic behavior of $b$), and prove it. Hint: Argue the number of $n$-bit Boolean functions that do not depend on some bit is at most $nc(n)$.

Problem 5. [10 points] You want to choose a team of $m$ people from a pool of $n$ people for your startup company, and from these $m$ people you want to choose $k$ to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m} \binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k} \binom{n-k}{m-k}.$$ 

Before you dump on your CFO or Harvard Business School, you decide to check his answer against yours — fortunately.

(a) Give an algebraic proof that your CFO’s formula agrees with yours.

(b) Now give what the Notes call a combinatorial argument proving this same fact.

Problem 6. [20 points] Suppose a generalized World Series between the Sox and the Cardinals involved $2n + 1$ games. As usual, the generalized Series will stop as soon as one team has won more than half the possible games.

(a) Suppose that when the Sox finally win the GSeries, the Cards have managed to win exactly $r$ games (so $r \leq n$). How many possible win-loss patterns are possible for the Sox to win the GSeries in this way? Express your answer as a binomial coefficient.

(b) How many possible win-loss patterns are possible for the Sox to win the GSeries when the Cards win at most $r$ games? Express your answer as a binomial coefficient.

(c) Give a combinatorial proof that

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}. \quad (1)$$
(d) Verify equation (1) by induction using algebra.

**Problem 7.** [15 points] In the following problem, you can leave your answers in closed form, that is, you do not have to write down the coefficients of the generating functions explicitly.

(a) Give a generating function for the number of ways to compose a bag of chocolate donuts, which contains at least 3 donuts.

(b) Give a generating function for the number of ways to compose a bag of glazed donuts, which contains at most 4 donuts.

(c) Give a generating function for the number of ways to compose a bag of coconut donuts, which contains exactly 2 donuts.

(d) Give a generating function for the number of ways to compose a bag of plain donuts, where the number of donuts in the bag is a multiple of 4.

(e) Give a generating function for the number of ways to compose a bag of donuts where the donuts are of 4 kinds (chocolate, glazed, coconut and plain) and all the following conditions hold:

- There are at least 3 are chocolate donuts, and
- at most 4 donuts are glazed. Also,
- there are 2 coconut donuts, and
- the number of plain donuts is a multiple of 4.

**Problem 8.** [20 points] A string of brackets is called “balanced” if the number of left brackets, [, equals the number of right brackets, ], and the number of ]’s never exceeds the number of [‘s as the string is read from left to right. For example,

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[ [ ] ] [ [ [ ] ] ]
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is a balanced string, but [ ] ] [ is not. There five different balanced strings with three ]‘s:

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[ [ [ ] ] ] [ [ ] ] [ [ ] ] ] [ [ ] ] [ ] [ ]
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Your diligent TA stayed up until dawn, slaving away with a dull pencil stub, working out
the following table:\footnote{This is a lie; he cranked it out with a C program in about six minutes.}

<table>
<thead>
<tr>
<th># [’s</th>
<th># of Balanced Strings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
</tr>
<tr>
<td>8</td>
<td>1430</td>
</tr>
<tr>
<td>9</td>
<td>4862</td>
</tr>
<tr>
<td>10</td>
<td>16796</td>
</tr>
<tr>
<td>11</td>
<td>58786</td>
</tr>
<tr>
<td>12</td>
<td>208012</td>
</tr>
</tbody>
</table>

In an apparently unrelated calculation, your exhausted TA then used his final drops
of energy and the last nubbin of his pencil to evaluate $500000 - 1000\sqrt{249999}$ out to 72
decimal places:\footnote{This is also a lie; he plugged the thing into Maple.}

$$1.000001000002000005000014000042000013200004290001430004862016796058786208012\ldots$$

Compare the table to the irrational number. The goal of this problem is to explain the
apparently miraculous correspondence.

(a) Let $p_n$ be the number of balanced strings containing $n$ [’s. Explain why the fol-
lowing recurrence holds:

\[ p_0 = 1, \quad (\text{the empty string}) \]

\[ p_n = \sum_{k=1}^{n} p_{k-1} \cdot p_{n-k}, \quad \text{for } n \geq 1. \]

(b) Let

\[ P(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots. \]

be the generating function for the number of balanced strings. Prove that

\[ P(x) = xP(x)^2 + 1. \]
(c) Find a closed-form expression for the generating function $P(x)$.

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

(e) Explain why the digits of this irrational number encode these successive numbers of balanced strings.