Problem Set 6

Due: Monday October 18 at 9 PM

Problem 1. A banker loans $100,000 to the Addams family so that they can purchase a new dilapidated mansion. At the end of each year for the next 20 years, Morticia Addams will drop by the bank to make a payment on the loan. The banker will immediately invest this returned money at a 7% annual rate of return.

What annual payment should the banker require of the Addams family so that his return after 20 years is the same as if he had invested the whole amount at 7% today instead of making the loan?

Problem 2. Find closed-form expressions for the following sums. Show your work.

(a) \[ \sum_{i=0}^{n} \frac{9^i - 7^i}{11^i} \]

(b) \[ \sum_{i=0}^{n} i^2 x^i \]

(c) \[ \sum_{i=0}^{n} \sum_{j=0}^{m} 3^i x^j \]

Problem 3. Pharaoh Aha I decides to build a “pyramid” in his honor consisting of a single block:

His successor, Aha II, trumps him by building a larger pyramid:
Problem 4. Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.) Show your work.

\[ \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \]

Problem 5. There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson stretches the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here’s what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now \(3 \cdot (3/2) = 4.5\) cm behind the bug and \(197 \cdot (3/2) = 295.5\) cm ahead.
- The bug walks another 1 cm in the third second, and so on.
Your job is to determine this poor bug’s fate.

(a) During second $i$, what fraction of the rug does the bug cross?

(b) Over the first $n$ seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic number $H_n$.

(c) Approximately how many seconds does the bug need to cross the entire rug?

Problem 6. Determine which of these choices

\[ \Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these} \]

describes each function’s asymptotic behavior. Proofs are not required, but briefly explain your answers.

(a) \[ n + \ln n + (\ln n)^2 \]

(b) \[ \frac{n^2 + 2n - 3}{n^2 - 7} \]

(c) \[ \sum_{i=0}^{n} 2^{2i+1} \]

(d) \[ \ln(n^2!) \]

(e) \[ \sum_{k=1}^{n} k \left( 1 - \frac{1}{2^k} \right) \]