Problem Set 4

Due: Monday, October 4 by 9PM

[Problem 6(a) slightly revised, 10/3, 10am.]

Problem 1. [10 points] The adjacency matrix of a graph is given below.

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\]

(a) Draw the graph defined by this adjacency matrix. Label the vertices of your graph \(1, 2, \ldots, 6\) so that vertex \(i\) corresponds to row and column \(i\) of the matrix.

(b) What is the diameter of this graph? Explain why.

(c) Find a cycle in this graph of maximum length and explain why it has maximum length.

(d) Give a coloring of the vertices that uses the minimum number of colors. Prove that this a minimum coloring.

Problem 2. [20 points] A property of a graph is said to be preserved under isomorphism if whenever \(G\) has that property, every graph isomorphic to \(G\) also has that property. For example, the property of having five vertices is preserved under isomorphism: if \(G\) has five vertices then every graph isomorphic to \(G\) also has five vertices.

In the following questions, you will asked to be prove whether certain pairs of graphs are isomorphic. If they are isomorphic, you can prove this by writing down an isomorphism. If they are not, you can prove this by giving a property \(p\) that is preserved under isomorphism such that one graph has property \(p\) but the other does not. You do not need to prove that your chosen property is preserved under isomorphism.

(a) Are the graphs \(G_1\) and \(G_2\) isomorphic?
(b) Are the graphs $G_1$ and $G_3$ isomorphic?

(c) Are the graphs $G_1$ and $G_4$ isomorphic?

Problem 3. [20 points] The following operations will be applied to a graph: pick any two vertices $u \neq v$.

1. If there is an edge of between $u$ and $v$ and there is also a path from $u$ to $v$ which does not include this edge, then delete the edge $\{u, v\}$.

2. If there is no path from $u$ to $v$, then add the edge $u \rightarrow v$.

Keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies. A graph to which no operation applies is called “stuck.”

(a) Let $G$ be the graph with vertices $\{1, 2, 3, 4\}$ and edges

\[
\{\{1, 2\}, \{3, 4\}\}.
\]
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Draw all the stuck graphs possible when starting with $G$.

(b) Prove that a graph that is stuck must be a tree.

(c) Let $e$ be the number of edges in $G$, let $c$ be the number of connected components it has. Show that the value of $3c + 2e$ gets smaller when either operation is applied. Conclude that no matter how you apply the operations to a graph, you will get stuck.

Problem 4. [20 points]

Theorem 1. A graph $G$ is bipartite if and only if it contains no odd cycle.

We’ll walk you through a proof. As is common with “if and only if” assertions, the proof has two parts. Part (a) asks you to prove that the left side of the “if and only if” implies the right side. Parts (b), (c) and (d) help you prove that the right side implies the left.

(a) Assume the left side and prove the right side. Three to five sentences should suffice.

(b) Explain why there is a 2-coloring of any tree, $T$.

(c) Suppose that $G$ is a connected graph with no odd cycle, and let $T$ be a spanning tree of $G$. Show that a 2-coloring of $T$ also defines a 2-coloring of $G$.

(d) Now just assume the right side, namely $G$ is a graph with no odd cycle, but is not necessarily connected. Describe a way of obtaining a 2-coloring of $G$.

Problem 5. [10 points] A perfect match in an arbitrary (not necessarily bipartite) graph is a subset of edges such that every vertex is an endpoint of exactly one edge in the subset. Informally, a perfect match corresponds to pairing up each person with a unique compatible buddy — who might be of the same sex — as opposed to the bipartite case in Hall’s Theorem, where matches must be to a person of the opposite sex.

Suppose a graph has two perfect matchings such that every edge is in at least one of the matchings. Prove that the graph is bipartite.

Hint: Observe that two edges incident on the same vertex cannot be in the same matching, and use the previous problem.

Problem 6. [20 points]

Suppose a connected, planar graph has $v$ vertices, $e$ edges, and $f$ faces.

(a) Show that if $v > 2$, then $e \leq 3v - 6$. 

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*Hint:* Every edge appears exactly twice on the borders of the faces of the graph\(^1\), and the border of every face has at least three appearances of edges. Assume these facts and use Euler’s formula.

(b) Show that the graph has a node of degree at most 5.

(c) Conclude that any planar graph can be colored with six colors.

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\(^1\)If an edge touches only one face, then it is considered to have two appearances on the border of the face (one “going” and the other “coming”). For example, if the whole graph consists of a single path with \(e\) edges, then it only has one face; the border of this face has \(2e\) appearances of edges.