Problem Set 10

Due: Monday November 22 at 9 PM

Problem 1. [10 points] We’re interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) What is an appropriate sample space to use for this problem? What are the outcomes in the event, \( E \), we are interested in? What are the probabilities of the individual outcomes in this sample space?

(b) What is \( \Pr(E) \)?

Problem 2. [15 points] Prove the following probabilistic identities. You may assume the theorem that the probability of a union of disjoint sets is the sum of their probabilities.

(a) \( \Pr(A - B) = \Pr(A) - \Pr(A \cap B) \).

(b) The Inclusion-Exclusion Rule for probability:

\[
\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).
\] (1)

(c) The Union Bound: Let \( A_1, \ldots, A_n \) be a collection of events. Then

\[
\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \Pr(A_i).
\]

Hint: Induction.

Problem 3. [15 points]

(a) Suppose we independently roll a fair die \( n \) times. What is the probability of our rolling a 4 at least once?

(b) Suppose we keep independently rolling a fair die until a 4 gets rolled. What is the probability that we roll exactly \( n \) times?
(c) Carefully define the probability space you used to answer part (b), and prove that your space satisfies the conditions required of a probability space.

(d) In the probability space you used in part (c), what is the set of outcomes in the event that a 4 never gets rolled? What is the probability of this event?

Problem 4. [10 points] There is a course—not 6.042, naturally—in which 10% of the assigned problems contain errors. If you ask a TA whether a problem has an error, then he or she will answer correctly 80% of the time. This 80% accuracy holds regardless of whether or not a problem has an error. Likewise when you ask a lecturer, but with only 75% accuracy.

Suppose you have doubts about a problem and ask a TA about it, and she tells you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that the correctness of the lecturers’ answer and the TA’s answer are independent of each other, regardless of whether there is an error.\textsuperscript{1} Now you want to figure out the probability that there is an error in the problem.

This question can be formulated as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

\[ E := \text{“the problem has an error,”} \]
\[ T := \text{“the TA says the problem has an error,”} \]
\[ L := \text{“the lecturer says the problem has an error.”} \]

(a) What are \( \Pr (E), \Pr (T \mid E), \text{and} \Pr (L \mid E)? \)

(b) Translate the independence assumptions into a couple of equations about \( \Pr (T \cap L \mid E) \) and \( \Pr (T \cap L \mid \bar{E}). \)

(c) Calculate \( \Pr (E \mid \bar{T} \cap L). \)

(d) Is the event that “the TA says that there is an error”, independent of the event that “the lecturer says that there is an error”? Explain.

Problem 5. [15 points] Three gunfighters, \( A, B, \) and \( C, \) plan to fight a pistol duel. They stand in a triangle so that they all have a clear shot at each other. All of them know that \( A’s \) chance of hitting his target is 0.3, \( B \) never misses, and \( C’s \) chance of hitting is 0.5. They will shoot one at a time in the sequence \( ABCABC \ldots \) (once a fighter is dead, he no longer gets a turn). On his turn, a gunfighter may choose to target either one of the other two, or may fire harmlessly into the air. Assume that each gunfighter uses the shooting strategy that maximizes his probability of being the only one left standing.

\textit{Note:} You may assume that the events of hitting the desired target at each turn are mutually independent.

\textsuperscript{1}This assumption is questionable: by and large, we would expect the lecturer and the TA’s to spot the same glaring errors and to be fooled by the same subtle ones.
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(a) Whenever it is B’s turn, what is his optimal strategy?

(b) What is A’s probability of surviving if:
   1. he misses on his first shot?
   2. he hits B on his first shot?
   3. he hits C on his first shot?

(c) Conclude that A’s best strategy is to fire in the air! What is his probability of survival?

Problem 6. [10 points] Most probability identities continue to hold when all probabilities are conditioned on the same positive probability event, C. For example, the Inclusion-Exclusion rule (1) can be conditioned on C to become the Conditional Inclusion-Exclusion Rule:

\[
\Pr(A \cup B \mid C) = \Pr(A \mid C) + \Pr(B \mid C) - \Pr(A \cap B \mid C).
\]  

(a) Define a new function, \( \Pr_C() \) by the rule that

Definition.

\[
\Pr_C(A) = \Pr(A \mid C).
\]

Prove that for any positive probability event, C, the function \( \Pr_C() \) is another probability measure (on the same sample space).

(b) Use part (a) to give a simple proof of the Sum Rule conditioned on C. Namely, suppose \( A \) is the disjoint union of sets \( A_1, A_2, \ldots, A_n \). Prove that

\[
\Pr(A \mid C) = \sum_{i}^{n} \Pr(A_i \mid C).
\]

Problem 7. [15 points]

(a) Suppose \( A \) and \( B \) are disjoint events. Prove that \( A \) and \( B \) are not independent, unless \( \Pr(A) \) or \( \Pr(B) \) is zero.

(b) If \( A \) and \( B \) are independent, prove that \( A \) and \( \overline{B} \) are also independent. You may find it useful to use results from Problem 2.

(c) Give an example of events \( A, B, C \) such that \( A \) is independent of \( B \), \( A \) is independent of \( C \), but \( A \) is not independent of \( B \cup C \).

(d) Prove that if \( C \) is independent of \( A \), and \( C \) is independent of \( B \), and \( C \) is independent of \( A \cap B \), then \( C \) is independent of \( A \cup B \). Hint: Calculate \( \Pr(A \cup B \mid C) \).
Problem 8. [10 points] Suppose there are 100 people in a room. Assume that their birthdays are mutually independent and uniformly distributed. Let $A$ be the event that two people have the same birthday. As stated in lecture notes, $\Pr(A) > 0.99$.

Suppose we fix a particular person in the class—call her “Jane”—and then ask everyone in the room except Jane when their birthday is. Let $B$ be the event that all of those 99 birthdays are different.

(a) What’s wrong with the following argument:

With probability greater than 0.99, some pair of people in the room have the same birthday. Since the 99 people we asked all had different birthdays, it follows that with probability greater than 0.99, Jane has the same birthday as some other person in the room.

(b) What is the actual probability that Jane has the same birthday as some other person in the room?