Things to Know - Compilation of Common Mistakes

1 Different Things To Know

You should know the following symbols and formulas. Ask your TA if you are not sure about a symbol or how to derive a formula.

1.1 Symbols and Notations

• Natural numbers: \( \mathbb{N} = \{0, 1, 2, \ldots \} \)
• Integers: \( \mathbb{Z} = \{0, 1, -1, 2, -2, \ldots \} \)
• Rational numbers or fractions: \( \mathbb{Q} = \{a/b \text{ s.t. } (a, b) \in \mathbb{Z} \times \mathbb{Z} \text{ and } b \neq 0\} \)
• Real numbers: \( \mathbb{R} = (-\infty, \infty) \)
• Complex numbers: \( \mathbb{C} = \{x + iy \text{ s.t. } x, y \in \mathbb{R} \text{ and } i^2 = -1\} \)
• Positive numbers \( \mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, \ldots \} \), \( \mathbb{R}^+ = (0, \infty) \)

• If \( S \) is a set, then \( |S| \) is the size (also called the cardinality) of \( S \). If \( r \) is a real number, then \( |r| \) is the absolute value of \( r \). If \( a \) is a complex number, then

\[
|a| = \sqrt{\text{real-part}(a)^2 + \text{complex-part}(a)^2}
\]

is called the modulus of \( a \).

1.2 Useful Formulas

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

\[
\forall r \neq 1 \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}
\]
2 Inaccurate Statements, False Proofs and Solutions

In the next section, you will see a compilation of inaccurate or false statements, as well as false proofs. Make sure that you can actually find the errors (mistakes or inaccuracies). Also, make sure you would be able to provide a correct solution. Solutions are provided at the end of the section.

2.1 Inaccurate Statements and False Proofs

1. \( a \mid b \Rightarrow b = ak \).
2. If \( \exists (a, b) \in \mathbb{Z}^2 \) such that \( a \mid b \) and \( b \mid a \), then \( a = b \).
3. \( \gcd(a, b) = sa + tb \).
4. \( x^2 \geq 0 \forall x \in \mathbb{R} \).
5. Prove the proposition \( Q : \gcd(ka, kb) = k \cdot \gcd(a, b) \).
   \textbf{Proof:} Since the gcd is the smallest linear combination of \( a \) and \( b \), \( \exists (s, t) \in \mathbb{Z}^2 \) such that \( \gcd(ka, kb) = s(ka) + t(kb) \). Similarly, \( \exists (s, t) \in \mathbb{Z}^2 \) such that \( \gcd(a, b) = sa + tb \). Therefore, we have: \( \gcd(ka, kb) = k \cdot \gcd(a, b) \).
6. Prove that \( \forall (x, y) \in \mathbb{R}^+ \sqrt{xy} \leq (x + y)/2 \).
   \textbf{Proof:} First Try! This is easy, since we have:
   \[
   \sqrt{xy} \leq \frac{x + y}{2}, \\
   xy \leq \frac{(x + y)^2}{4}, \\
   4xy \leq x^2 + 2xy + y^2, \\
   0 \leq x^2 - 2xy + y^2, \\
   0 \leq (x - y)^2.
   \]
   Since \( 0 \leq (x - y)^2 \), we have \( \sqrt{xy} \leq (x + y)/2 \).
7. Prove that \( \forall (x, y) \in \mathbb{R}^+ \sqrt{xy} \leq (x + y)/2 \).
   \textbf{Proof:} Second Try! Let’s try to be more careful and let’s use the obvious true fact that:
   \( \forall (x, y) \in \mathbb{R}^2, 0 \leq (x - y)^2 \).
We have:

$$\forall (x, y) \in \mathbb{R}^2, \ 0 \leq (x - y)^2.$$  
$$\Rightarrow \forall (x, y) \in \mathbb{R}^2, \ 0 \leq x^2 - 2xy + y^2 \ (\text{since} \ (x - y)^2 = x^2 - 2xy + y^2)$$  
$$\Rightarrow \forall (x, y) \in \mathbb{R}^2, \ 4xy \leq x^2 + 2xy + y^2 \ (\text{adding} \ 4xy \ \text{on each side})$$  
$$\Rightarrow \forall (x, y) \in \mathbb{R}^2, \ 2\sqrt{xy} \leq (x + y) \ (\text{taking} \ \text{the square root on each side})$$  
$$\Rightarrow \forall (x, y) \in \mathbb{R}^2, \ \sqrt{xy} \leq \frac{x + y}{2} \ (\text{dividing by} \ 2).$$

Since we worked with implications, the proof is established.

8. Prove that for $c \neq 0$, $a \mid b \iff ca \mid cb$.

**Proof**: We can directly prove the equivalence of both propositions. We have: $a \mid b$  
$\iff \exists k \in \mathbb{Z} \ \text{such that} \ b = ka \iff \exists k \in \mathbb{Z} \ \text{such that} \ cb = kca \iff ca \mid cb$, and the proof is completed.

9. Prove by induction that for any natural number $n$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}$$

**Proof**: first try! We prove by induction.

Induction Hypothesis: Let $P(n)$ be

$$\forall n, \ 1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}$$

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10. Prove that $\forall n \in \mathbb{N}^+ \frac{2}{n(n+1)(n+2)} = \frac{1}{n} \frac{2}{n+1} + \frac{1}{n+2}$.

**Proof**: We have:

$$\forall n \in \mathbb{N}^+ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \ \text{and} \ \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

Also, we have the following equality:

$$\forall n \in \mathbb{N}^+ \frac{2}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}.$$ 

Therefore, we can write that

$$\forall n \in \mathbb{N}^+ \frac{2}{n(n+1)(n+2)} \iff \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$$

$$\iff \frac{1}{n} - \frac{1}{n+1} - \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$\iff \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}.$$
which proves the equality.

11. Prove by induction that for any natural number $n$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}$$

**Proof**: second Try!

- We prove by induction.
- Induction Hypothesis: Let $P(n)$ be

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}$$

- Base case: we have $0 = \frac{0(0+1)}{2}$, so $P(0)$ is true.
- Inductive step: let’s assume that $P(n)$ is true for some natural number $n$. We have:

$$P(n + 1) = 1 + 2 + \cdots + n + (n + 1)$$

$$\Rightarrow P(n + 1) = \frac{n(n+1)}{2} + (n + 1) \quad \text{(since we assume $P(n)$ is true)}$$

$$\Rightarrow P(n + 1) = 1 + 2 + \cdots + n + (n + 1) = \frac{(n+1)(n+2)}{2}$$

- Therefore, we have proven by induction that for any natural number $n$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}$$

## 2.2 Solutions

1. You need to specify what $k$ is. The previous statement is inaccurate and, as a consequence, should be considered as false. Correct statements are:

- $a \mid b \iff b = ak$ for some integer $k$, or
- $a \mid b \iff b = ak$ for some $k \in \mathbb{Z}$, or
- $a \mid b \iff \exists k \in \mathbb{Z}$ such that $b = ak$.

Note that you should not use quantifiers on the right side of the expression you want to quantify.

2. Don’t forget that the case $a = -b$ is possible too. These mistakes are often resulting from lack of attention. Just be careful. Two correct statements would be:
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- If \( \exists (a, b) \in \mathbb{Z}^2 \) such that \( a \mid b \) and \( b \mid a \), then \( a = b \) or \( a = -b \), or
- If \( \exists (a, b) \in \mathbb{N}^2 \) such that \( a \mid b \) and \( b \mid a \), then \( a = b \).

3. Once again, you need to specify what \( s \) and \( t \) are. Correct statements could be:
   - \( \exists (s, t) \in \mathbb{Z}^2 \) such that \( \gcd(a, b) = sa + tb \), or
   - \( \gcd(a, b) = sa + tb \) for some integers \( s \) and \( t \).

4. As previously mentioned, you should not use quantifiers on the right side of the expression you want to quantify. Correct statements should be:
   - \( x^2 \geq 0 \) for all \( x \in \mathbb{R} \), or
   - \( x^2 \geq 0 \) for all real numbers \( x \), or
   - \( \forall x \in \mathbb{R}, \ x^2 \geq 0. \)

5. There are two mistakes in this proof.
   - The first one is that the \( \gcd \) of \( a \) and \( b \) is not the smallest linear combination of \( a \) and \( b \). The \( \gcd \) of \( a \) and \( b \) is the smallest positive linear combination of \( a \) and \( b \). Forgetting positive is inaccurate and, as a consequence, false.
   - The second one is more subtle but as important if not more! When writing:
     - \( P1 : \exists (s, t) \in \mathbb{Z}^2 \) such that \( \gcd(ka, kb) = s(ka) + t(kb) = k(sa + tb) \) and
     - \( P2 : \exists (s, t) \in \mathbb{Z}^2 \) such that \( \gcd(a, b) = sa + tb \),

   the integers \( s \) and \( t \) in \( P1 \) do not have to be the same as the integers \( s \) and \( t \) in \( P2 \). Make sure you understand this point! If you had written \( P2 \) as \( P2 : \exists (p, q) \in \mathbb{Z}^2 \) such that \( \gcd(a, b) = pa + qb \), this fact would have been more obvious. Therefore, proving that \( \gcd(ka, kb) = k \cdot \gcd(a, b) \) by writing that: \( \gcd(ka, kb) = s(ka) + t(kb) = k(sa + tb) = k \cdot \gcd(a, b) \) is completely wrong!

6. The proof is what you could call a backward proof. The way the proof is written does not prove anything! The proof shows that \( Q \Rightarrow \text{True} \), which does not imply that \( Q \) is true! I hope that you clearly understand this point and that you remember the truth table of \( P \Rightarrow Q \): if \( P \) is false and \( Q \) is true or false, \( P \Rightarrow Q \) is always true. Therefore, showing that \( Q \Rightarrow \text{True} \) does not reveal any information about \( Q \). The two ways to prove this fact would be directly \( \text{True} \Rightarrow Q \) or by contradiction \( \neg Q \Rightarrow \text{False} \). On top of the conceptual mistake, \( x \) and \( y \) are not quantified: we don’t know what they are; are they natural numbers, fractions, integers, real numbers, others. A good proof would specify that \( (x, y) \) are positive real numbers.

7. Once again, there is a small but important mistake in this proof. If your proof were correct, this would mean that you would have proven that \( 1 \leq -1 \) (take \( x = -1 \) and \( y = -1 \) for instance). So where is the problem then? In the written proof \( x \) and \( y \) are assumed to be real numbers, which means that they could be negative numbers. But if \( x < 0 \), then \( \sqrt{x^2} = -x \). This means that you have to be careful about what
you have to prove. The expression has to be proven for positive real numbers, so a correct proof should be:

\[ \forall (x, y) \in \mathbb{R}^+, 0 \leq (x - y)^2. \]

\[ \Rightarrow \forall (x, y) \in \mathbb{R}^+, 0 \leq x^2 - 2xy + y^2 \quad \text{(since } (x - y)^2 = x^2 - 2xy + y^2) \]

\[ \Rightarrow \forall (x, y) \in \mathbb{R}^+, 4xy \leq x^2 + 2xy + y^2 \quad \text{(adding } 4xy \text{ on each side)} \]

\[ \Rightarrow \forall (x, y) \in \mathbb{R}^+, (x + y)^2 \leq (x - y)^2 \quad \text{(since } (x - y)^2 = x^2 + 2xy + y^2) \]

\[ \Rightarrow \forall (x, y) \in \mathbb{R}^+, 2\sqrt{xy} \leq (x + y) \quad \text{(taking the square root on each side)} \]

\[ \Rightarrow \forall (x, y) \in \mathbb{R}^+, \sqrt{xy} \leq \frac{x + y}{2} \quad \text{(dividing by } 2). \]

Another way to prove it would be by equivalence:

\[ \forall (x, y) \in \mathbb{R}^+, \sqrt{xy} \leq \frac{x + y}{2} \]

\[ \Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2, 2\sqrt{xy} \leq (x + y) \]

\[ \Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2, 4xy \leq (x + y)^2 \]

\[ \Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2, 4xy \leq x^2 + 2xy + y^2 \]

\[ \Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2, 0 \leq x^2 - 2xy + y^2 \]

\[ \Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2, 0 \leq (x - y)^2. \]

Since we worked by equivalence, and the last proposition is true, the initial proposition is true. Therefore, we have proven the proposition. Working by equivalence is correct as long as you make sure that everything is really equivalent.

8. Once again the proof is incorrectly written. The proposition

\[ P_1 : \Leftrightarrow \exists k \in \mathbb{Z} \text{ such that } b = ka \]

is not equivalent to

\[ P_2 : \Leftrightarrow \exists k \in \mathbb{Z} \text{ such that } cb = kca \]

unless \( c \neq 0 \) is explicitly written! You have

\[ \forall c \in \mathbb{R}, P_1 \Rightarrow P_2, \]

but what you need in this proof is

\[ \forall c \in \mathbb{R}, c \neq 0 \Rightarrow P_1 \Leftrightarrow P_2. \]

If the quantifier \( \forall c \in \mathbb{R} \) is forgotten in your proof, the proof becomes false! A correct proof would be:

\[ a \mid b \Leftrightarrow \exists k \in \mathbb{Z} \text{ such that } b = ka \]

\[ \Leftrightarrow \exists k \in \mathbb{Z} \text{ s.t. } cb = kca, \text{ since } c \neq 0 \]

\[ \Leftrightarrow ca \mid cb, \]

and the proof is completed.
9. The induction hypothesis should be a predicate with \( n \) being the free variable. So it should not contain the quantifier \( \forall n \). In other words, the prove should go as follows:

**Proof:** We prove by induction.

Induction Hypothesis: Let \( P(n) \) be

\[
1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}.
\]

10. The equivalence symbol ’\( \Leftrightarrow \)’ should be used to state the equivalence of propositions only. It does not make sense to use ’\( \Leftrightarrow \)’ instead of ’\( = \)’ in a proof. The proof should go as follow:

**Proof:** We have:

\[
\forall n \in \mathbb{N}^+ \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}
\]

and

\[
\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}
\]

Also, we have the following equality:

\[
\forall n \in \mathbb{N}^+ \quad \frac{2}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}
\]

Therefore, we can write that:

\[
\forall n \in \mathbb{N}^+ \quad \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{1}{n+1} - \left( \frac{1}{n+1} - \frac{1}{n+2} \right)
\]

\[
\Leftrightarrow \forall n \in \mathbb{N}^+ \quad \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}
\]

which proves the equality.

11. If \( P(n) \) is defined as the predicate of the induction proof, \( P(n) \) can only be true or false. Therefore, you should not use \( P(n) \) in an equality with numbers. In other words,

\[
P(n+1) = 1 + 2 + \cdots + n + (n+1)
\]

does not make any sense. The proof should go as follows:

**Proof:**

- We prove by induction.
- Induction Hypothesis: Let \( P(n) \) be

\[
1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n+1)}{2}
\]
Let’s call $S(n)$ the following sum:

$$S(n) = 1 + 2 + \cdots + n$$

- Base case: we have $S(0) = 0 = \frac{0(0+1)}{2}$, so $P(0)$ is true.
- Inductive step: let’s assume that $P(n)$ is true for some natural number $n$. We have:

$$S(n + 1) = 1 + 2 + \cdots + n + (n + 1)$$

$$\Rightarrow S(n + 1) = \frac{n(n + 1)}{2} + (n + 1) \quad \text{(since we assume } P(n) \text{ is true)}$$

$$\Rightarrow S(n + 1) = 1 + 2 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

- Therefore, we have proven by induction that for any natural number $n$,

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} \frac{n(n + 1)}{2}$$