Lecture Review

Lecture 23: Infinity and Diagonalization

- If $A,B$ are sets, then $|A|, |B|$ denote the “sizes” of $A$ and $B$. We extend our intuition from finite sets to all sets and say $|A| = |B|$ if and only if there is a bijection between $A$ and $B$.

- A set is countable if there is a bijection between it and the natural numbers. An infinite set which is not countable is said to be uncountable. $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{Z} \times \mathbb{Z}$ are all countable, but $\mathbb{R}$ is not.

- Understand diagonalization as a proof technique and know how to use it to show (i) $\mathbb{R}$ is uncountable, (ii) the cardinality of the power set of any set is always bigger than the cardinality of the original set.

- Know that there are many different infinities, the smallest of which is $\aleph_0$ (the cardinality of the naturals), followed by $\aleph_1, \aleph_2$, etc. The cardinality of the reals is $2^{\aleph_0}$, but it is not provable what the relation of this is to $\aleph_1, \aleph_2$, etc.

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In Class Examples

Finite Subsets

Let $A = \{ X \subset \mathbb{N} : |X| \text{ is finite } \}$. Prove that $A$ is countable.

Note the difference between this and the fact proved in class that the power set of $\mathbb{N}$ is uncountable. As an interesting aside, the set of all subsets of an infinite set which have cardinality strictly less than the original set has the same cardinality as the original set.

A Strange Set

As you saw in lecture, many strange things can happen when we look at infinity. This example is known as Cantor’s Middle Third set and at first defies all intuition. We construct Cantor’s Middle Third set as follows: Start with the interval $[0,1]$ and remove the middle third of it, so we are left with $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now remove the middle third of each of these intervals leaving us with $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Now remove the middle third of each of those intervals, etc, etc.

(a) Prove that the total length of all the intervals removed is 1.

(b) Prove that despite this, the Cantor set is still uncountable and can be put in bijection with the interval $[0,1]$. 

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*This is very different than saying it is unknown. It has been proved that you can’t prove or disprove that fact that $2^{\aleph_0} = \aleph_1$.*
Group Problems

Infinities

We showed in class that $[0, 1]$ is uncountable, but what about all of $\mathbb{R}$? Could it be bigger? It turns out that it isn't, and the following steps should lead you to that conclusion:

(a) Give a bijection between $(0, 1)$ and $[0, \infty)

(b) Extend part (a) to give a bijection between $(0, 1)$ and all of $\mathbb{R}$

The Hotel Infinity

Note: Throughout this problem, infinity means countably infinite.

After years of hard labor, Luis is finally ready to open his brand new hotel: the Hotel Infinity. Now this hotel is rather unique as it has an infinite number of rooms, one for each the natural numbers, in fact. Obviously such a hotel is quite the tourist attraction and filled up almost immediately after opening. Yes, that’s right—there is currently a guest in each and every room of the hotel. This is where you come in. As the desk clerk, it is your job to make sure anybody who ever wants a room gets one, no matter how many other guests are already checked in. In fact, if you ever turn away a single guest, Luis has promised that he will fire you! In order to make your job slightly easier, each room is equipped with a loudspeaker, and guests are told upon checkin that they may be required to change rooms during their stay. How would you deal with the following situations?

• $k$ new people show up asking for rooms, where $k \in \mathbb{N}$.

• A tour bus filled with infinitely many excited tourists shows up.

• Infinitely many tour buses show up, each filled with infinitely many excited tourists.

The Limits of Induction

Induction is a great tool for proving that some statement is true for all natural numbers, but unfortunately it fails us when we want to prove something holds in the infinite case\(^1\). This problem is an example of one such failure.

• Prove by induction that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ is finite for any $n \in \mathbb{N}$.

• Show that there is a bijection between $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ and the interval $[0, 1]$ and thus this set is not countably many

\(^1\)This is a blatant lie—there is a form of induction called transfinite induction that works for infinity, but it is beyond the scope of this class. See MathWorld or any introductory analysis textbook for more information.